A COUNTING FORMULA FOR EDGE RECONSTRUCTION OF GRAPHS

By

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A COUNTING FORMULA FOR EDGE RECONSTRUCTION OF GRAPHS

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The edge reconstruction conjecture states that a graph G is uniquely determined, up to isomorphism, by $\{G - e \mid e \in E(G)\}$, the collection of the edge-deleted subgraphs of G. A new counting approach to edge reconstruction is introduced in this thesis. Using the counting formula, it is proved that a 2-edge-connected bipartite graph $G = (A, B)$ is edge reconstructible if $\varepsilon(G) > \frac{1}{2}\nu(A)\nu(B)$, where $\varepsilon(G)$, $\nu(A)$ and $\nu(B)$ denote the number of edges in G, the number of vertices in A and the the number of vertices in B, respectively, and that a separable graph G with 3-connected pruned center P is edge reconstructible if $\varepsilon(P) > \binom{p}{2}$, where $p = \nu(P)$. The main and most difficult result is that a graph G with pruned center P is edge reconstructible if $\varepsilon(P) > \frac{1}{4}p(p+3)$. In proving this theorem a result of Greenwell and Hemminger on the reconstructibility of the set of branches of a separable graph is improved.

CHAPTER 1
INTRODUCTION

A graph G is a pair $(V(G), E(G))$, where $V(G)$ is called the *vertex set* of G and $E(G) \subseteq V(G) \times V(G)$ is called the *edge set* of G. If $(u, v) \in E(G)$, then u and v are said to be joined by an edge or *adjacent*. The graph G is *finite* if both the vertex set and the edge set are finite. G is *undirected* if each pair (u, v) in $E(G)$ is unordered. If each pair (u, v) is ordered, then G is called a *directed graph* or a *digraph*. A graph G without loops or multiple edges is called *simple*. All graphs in this paper are finite, simple and undirected unless stated otherwise. Let $\nu(G) = |V(G)|$ and $\varepsilon(G) = |E(G)|$. The number of edges adjacent to v in G is called the *degree* of v in G and is denoted $d_G(v)$. A vertex v is called a *k-vertex* if $d_G(v) = k$. An edge adjacent to a 1-vertex is called a *pendant edge*. The *degree sequence of a graph* G is the sequence obtained by listing the degrees of its vertices in nondecreasing order. A graph G is called *k-regular* if all its vertices have degree k. A k-regular graph is simply called a *regular graph* when the value k is not emphasized. A *complete graph* is a graph such that each pair of its vertices are adjacent. A graph G is *disconnected* if the vertex set of G can be partitioned into two non-empty sets V_1 and V_2 such that there is no edge between a vertex V_1 and a vertex of V_2. If G is not disconnected, it is called *connected*. A graph G is called *separable* if G is connected and there is a vertex $v \in V(G)$ such that $G - v$ is disconnected, where $G - v$ is a subgraph of G obtained by deleting v and all its incident edges. The vertex v is called a *cutpoint* of G. For a separable graph G which is not a tree, define the *pruned graph* of G, *pruned*(G), to be the maximal subgraph of G that has no 1-vertex. A maximal connected subgraph that has exactly one vertex in common with *pruned*(G) is called a ***tree-growth*** in G. A 2-***block*** in a graph is a maximal 2-connected subgraph. Notice that a 2-block could be an edge, in which case it is called a ***trivial* 2-*block***. The ***block-cutpoint tree*** of a graph G, *block*(G), has the set of all cutpoints and all 2-blocks of G as its vertex set. Two vertices of *block*(G) are joined by an edge if and only if one of them is a 2-block of G and the other is a cutpoint of G on the 2-block. For any block-cutpoint tree T, it can be easily shown that there exists a unique vertex in T such that the distances between this vertex and any two endpoints of any longest path are always the same. This unique vertex is called the ceter of T. The 2-block or cutpoint P of G which is the center of the tree *block*(*pruned*(G)) is called the ***center*** of G. A ***branch*** B of G is a maximal subgraph of G that contains exactly one vertex u of the pruned center P, called the *root* of B, such that $B - u$ is connected. A branch B is called a ***rooted branch*** if the root of B is labeled. (For example, the root can be labeled by coloring it blue.) Hereafter, branch will always mean rooted branch. The set of all branches of G is denoted by β_G, or simply β if G is clear from the context.

Let $\{G-v\} = \{G-v \,|\, v \in V(G)\}$ be the collection of all vertex-deleted subgraphs of G. A *reconstruction* of G is a graph H such that $\{G-v\} = \{H-u\}$, that is, there is a bijection f between $V(G)$ and $V(H)$ such that $G - v \cong H - f(v)$, $\forall v \in V(G)$. A property or parameter P is *reconstructible* if, for any graph G with the property or parameter P, all its reconstructions also have the property or parameter P. For example, being disconnected is a reconstructible property; the degree sequence is a reconstructible parameter [1]. A graph G is *reconstructible* if, for any reconstruction H of G, we have $H \cong G$. Shown in Figure 1.1 are a pair of graphs G and H that are not reconstructible.

Reconstruction Conjecture. All finite, undirected, simple graphs with at least three vertices are reconstructible.

The example in Figure 1.1 shows the necessity of at least three vertices. The Reconstruction Conjecture, posed in 1941 by Kelly and Ulam [2], is regarded as one of the foremost unsolved problems in graph theory. The original form of the problem can be found in Ulam's book "A collection of mathematical problems" [3, p.29]. It was stated as an algebraic problem, as follows.

The original version of reconstruction problem. "Suppose that in two sets A and B, each of n elements, there is defined a distance function ρ for every pair of distinct points, with values either 1 or 2, and $\rho(p, p) = 0$. Assume that for every subset of $n - 1$ points of A, there exists an isometric system of $n - 1$ points of B, and the number of distinct subsets isometric to any given subset of $n - 1$ points is the same in A as in B. Are A and B isometric ? "

The original problem can be easily transformed to the graph reconstruction problem by defining two graphs G and H with the vertex set A and B, respectively, such that for any two vertices u and v, (u, v) is an edge if and only if $\rho(u, v) = 1$.

The first published work on this conjecture is due to P.J. Kelly. He proved in 1957 that the conjecture is true for trees, disconnected graphs and regular graphs [4]. To prove these results, Kelly established a fundamental lemma.

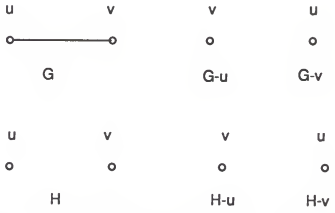

Figure 1.1. Here we have $\{G - v\} = \{H - v\}$, but $H \ncong G$.

Kelley's Lemma. For any two graphs F and G such that $\nu(F) < \nu(G)$, the number $s(F,G)$ of subgraphs of G isomorphic to F is reconstructible.

Actually, he proved that

$$s(F,G) = \sum_{v \in V(G)} \frac{s(F, G-v)}{\nu(G) - \nu(F)},$$

which implies Kelly's Lemma. Since then, much work has been done on the Reconstruction Conjecture. It has been checked that the conjecture is true for graphs with at most 9 vertices [5, p. 229]. Bondy showed in 1969 [6] that separable graphs with no $1 - vertex$ are reconstructible. In the same year, Greenwell and Hemminger [7] proved that, for a separable graph G, the set β_G is reconstructible if there is a 1-vertex v such that $G - v$ has at least two branches with 1-vertices. Also in 1969, Geller and Manvel [8] generalized Kelly's result on trees and proved that Cacti (a *cactus* is a graph whose non-trivial 2-blocks are all cycles) are reconstructible. Their result was later extended by this author to the graphs whose non-trivial 2-blocks are all Eulerian graphs (a graph is called an *Eulerian graph* if all its vertices have even degrees) [9]. Most of the results obtained so far are on separable graphs. It seems that non-separable graphs (that is, 2-connected graphs) are much more difficult to deal with. In fact, this author proved in 1988 that if all 2-connected graphs are reconstructible, then the Reconstruction Conjecture is true [10].

During the past half century, many problems related to the Reconstruction Conjecture have been proposed. These problems, together with the Reconstruction Conjecture, form a special area in the study of graph theory, called graph reconstruction problems. Some of these problems are listed below.

Problem 1. Reconstruction of digraphs.

The reconstruction problem for digraphs is defined similar to undirected graphs. The Reconstruction Conjecture is false for digraphs. In fact, it is even not true for tournaments, the digraphs obtained by assigning a direction to each edge of a complete graph. Moreover, Stockmeyer (1976, [11]) constructed non-reconstructible tournaments on $2^n + 2^m$ vertices for all m and n, not both zero, thus dashing any hope of modifying the Reconstruction Conjecture by putting a restriction on the number of vertices. However, a modified version of the digraph reconstruction problem, called the N-reconstruction problem (see below), has been proposed and is still open as of today.

Let $d_{out}(u)$, called *out-degree* of the vertex u of a digraph, be the number of arcs of the form (u, w) in D and let $d_{in}(u)$, called *in-degree* of the vertex u, be the number of arcs of the form (w, u) in D. $(d_{out}(u), d_{in}(u))$ is called the *degree-pair* of u. In 1981, Ramachandran [12] defined a digraph D to be N-reconstructible if it is reconstructible from $\{D - v | v \in V(D)\}$ provided the degree-pair of the deleted vertex is supplied with each vertex-deleted subgraph in $\{D - v | v \in V(D)\}$. He pointed out that all counterexamples to digraph reconstruction are N-reconstructible [13], thus raised the following problem.

N-Reconstruction Conjecture. All digraphs are N-reconstructible.

Problem 2. Reconstruction of infinite graphs.

A graph G is called an infinite graph if either $V(G)$ or $E(G)$ is an infinite set. The reconstruction problem for infinite graphs is stated similarly to the case of finite graphs [14]. There are many counterexamples to the infinite graph reconstruction problem. The simplest one is a pair of graphs T_∞ (the $\aleph_0$-regular tree) and $2T_\infty$ [15]. However, there is another, perhaps more natural, version of the infinite graph reconstruction problem, proposed by R. Halin (see Bondy and Hemminger's paper, [16]). Halin observed that all known non-reconstructible pairs have the property that each is isomorphic to an induced subgraph of the other. The graphs with this property are called *H-isomorphic*. A graph G is *H-reconstructible* if every reconstruction of G

is H-isomorphic to G. Notice that in case of a finite graph, H-reconstructible and reconstructible are equivalent. The following conjecture is still open.

Halin's Conjecture. All graphs are H-reconstructible.

Among all the reconstruction problems, the one that has received most attention in recent years is the edge reconstruction problem, posed by Harary in 1964 [14].

Problem 3. Edge reconstruction problem.

Let G be a graph. For all $e \in E(G)$, define the edge deleted subgraph $G-e$ to be the subgraph of G obtained by deleting the edge e from G. Denote $\{G-e\} = \{G-e \,|\, e \in E(G)\}$. An *edge reconstruction* of G is a graph H such that $\{H-e\} = \{G-e\}$. A property or parameter of a graph being edge reconstructible is defined similarly to the vertex version. A graph G is *edge reconstructible* if, for any edge reconstruction H of G, we have $H \cong G$. Shown in Figure 1.2 are a pair of graphs G and H that are not edge reconstructible.

Edge Reconstruction Conjecture. All finite, undirected, simple graphs with at least four edges are edge reconstructible.

The example in Figure 1.2 shows the necessity of at least four edges. Progress on the Edge Reconstruction Conjecture has been steady. In 1971, Greenwell showed that for any graph G (with at least four edges and without any isolated vertex), the set $\{G-e\}$ determines the set $\{G-v\}$ [17]. This implies that all reconstructible

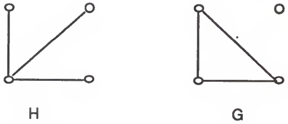

Figure 1.2. Here we have $\{G-e\} = \{H-e\}$, but $H \ncong G$.

properties or parameters are also edge reconstructible. In particular, all vertex reconstructible graphs mentioned thus far are also edge reconstructible. Much recent research has been inspired by an important paper of Lovász published in 1972 [18], in which he proved that a graph G is edge reconstructible if $\varepsilon(G) > \frac{1}{2}\binom{\nu(G)}{2}$. In 1977 Müller [19] extended this result by proving that a graph G is edge reconstructible if $2^{\varepsilon(G)-1} > \nu(G)!$. In 1987, Ellingham, Hoffman and Myrvold [20] proved that bi-degree graphs are edge reconstructible. In 1988, Ellingham, Pyber and Yu [21] showed that $K_{1,3}$−free graphs are edge reconstructible.

In Section 2.1 that follows, a new counting formula is proved. Unlike the counting formulas used by Lovász or Müller, which require that the whole graph have sufficiently many edges, the new counting formula only requires that a certain subgraph have sufficiently many edges. In section 2.2, our counting formula is used to prove that a 2-edge-connected bipartite graph $G = (A, B)$ is edge reconstructible if $\varepsilon(G) > \frac{1}{2}\nu(A)\nu(B)$ and that a separable graph G with 3-connected pruned center P is edge reconstructible if $\varepsilon(P) > \frac{1}{2}\binom{\nu(P)}{2}$. In Chapter 3, the assumption that P is 3-connected is dropped in the latter result. It is proved that a graph G with pruned center P is edge reconstructible if $\varepsilon(P) > \frac{1}{4}p(p+3)$. To obtain this result, it is proved in Section 3.2 that the set of branches of a separable graphs G is reconstructible unless all the following hold: (1) G has exactly two branches, (2) one branch contains all the 1-vertices of G, (3) the other branch contains exactly one end-block, and (4) the pruned center of G is a vertex or an edge. This is an improvement on the previously mentioned theorem of Greenwell and Hemminger [7].

CHAPTER 2
NEW COUNTING FORMULA

2.1 The Counting Formula

In this section a counting formula for edge reconstruction is given. The graph G is regarded as a subgraph of K_ν. Hereafter S, T is a partition of the edge set of the complete graph K_ν such that S is either a complete subgraph of order not more than ν or a complete bipartite subgraph of order ν. The later case occurs only if G itself is bipartite, in which case the bipartition of G and S must be the same. Let $Q_G = G \cap S$ and $\overline{Q}_G = \overline{G} \cap S$. If G is understood, Q_G and $\overline{Q}_G$ are simply denoted Q and $\overline{Q}$, respectively. Note that $\overline{Q}$ is the complement of Q *in* S. A subgraph of K_ν is called *full* if it contains all edges of $G \cap T$, possibly some edges of S and no other edges. Given a full graph F, a *copy* F' of F is a full subgraph of K_ν such that $F' \cong F$ with $G \cap T$ corresponding to $G \cap T$ under the isomorphism. A copy F' of F is called a *true G-copy* if $F' \subseteq G$. Let $s(F, G)$ denote the number of true G-copies of F. If G is understood, a true G-copy is simply called a *true copy.* A copy of F that is not a true copy is called a *fake G-copy,* or simply a *fake copy* when G is understood.

Denote $\tilde{F} = F - (G \cap T)$. Note that $\tilde{F} = F \cap Q$ when $F \subseteq G$. For a given full graph F, define $t_G(F)$ to be the total number of copies of F counted in $\{G+e \mid e \in \overline{Q}\}$, that is,

$$t_G(F) = \sum_{e \in \overline{Q}} s(F, G+e).$$

Let $[F-e]$ be the maximum subset of $\{F - e \mid e \in \tilde{F}\}$ such that no two graphs in $[F-e]$ are isomorphic to each other. In another words, if the set $\{F - e \mid e \in \tilde{F}\}$ is partitioned into isomorphism classes, then $[F-e]$ is a set of representatives of these classes. Given $F-e \in [F-e]$, the number of ways to add an edge of S to any true copy of $F-e$ to get a copy of F is denoted by $a(F-e)$. We now show that $a(F-e)$ is independent of the embedding of $F-e$ in G. Consider separately the cases S complete and S complete bipartite. Suppose that S is a complete subgraph. Let $(F-e)'$ and $(F-e)''$ be two true copies of $F-e$ in G. By the definition of being a copy, there must be an isomorphism $f : (F-e)' \cong (F-e)''$ such that $f(G \cap T) = G \cap T$ and hence f takes vertices of S to vertices of S. Hence adding an edge $(u,v) \in S$ to $(F-e)'$ yields a (true or fake) copy of F if and only if adding $(f(u), f(v)) \in S$ to $(F-e)''$ yields a (true or fake) copy of F. Therefore $a(F-e)' = a(F-e)''$. Suppose now that S is a complete bipartite graph. Let the bipartition of S (and of G) be U, V. Let $(F-e)'$ and $(F-e)''$ be two true copies of $F-e$ in G. By the definition of being a copy, they must both contain $G \cap T$, that is contain all edges of two complete subgraphs induced by vertices of U and V, respectively. Moreover, there is an isomorphism $f : (F-e)' \cong (F-e)''$ with $G \cap T$ corresponding to $G \cap T$. For any two vertices $u \in U$ and $v \in V$, if $f(u)$ and $f(v)$ are both in U or both in V, then one of u or v must be adjacent to all vertices in $(F-e)'$, in particular, $(u,v) \in (F-e)'$. Hence for any two vertices $s \in U$ and $t \in V$, if $(s,t) \notin (F-e)'$, then $f(s)$ and $f(t)$ cannot both be in U or both in V. Clearly adding (s,t) to $(F-e)'$ yields a copy of F if and only if adding $(f(s), f(t))$ to $(F-e)''$ yields a copy of F. Moreover, since $f(s)$ and $f(t)$ are not both in U or both in V, we have $(f(s), f(t)) \in S$. Hence $a(F-e)' = a(F-e)''$.

Theorem 2.1 (Counting Formula). If F is full and $\varepsilon(\tilde{F}) \neq \varepsilon(\overline{Q})$, then

$$s(F,G) = \frac{[\sum_{F-e\in[F-e]} s(F-e,G)a(F-e)] - t_G(F)}{\varepsilon(\tilde{F}) - \varepsilon(\overline{Q})} \tag{1}$$

Proof. Define $t' = t'_G(F)$ to be the total number of fake G-copies of F counted in $\{G + e \mid e \in \overline{Q}\}$. Compute t' as follows. A fake copy of F has the form $(F - e)' + f$, where $(F - e)' \subseteq G$ is a true copy of some $F - e \in [F - e]$ and $f \in \overline{Q}$. For a particular subgraph $(F - e)'$ of G, there are $a(F - e)$ ways to add an edge to get a copy of F, that is, there are $a(F - e)$ edges f in S such that $(F - e)' + f$ is a copy of F. Denote by $b(F - e)'$ the number of such f in Q. Then there are exactly $a(F - e) - b(F - e)'$ fake copies of F having the form $(F - e)' + f$. Hence

$$t' = \sum_{F-e \in [F-e]} \; \sum_{(F-e)' \cong F-e} [a(F - e) - b(F - e)'].$$

The number of true copies $(F - e)'$ isomorphic to $F - e$ is $s(F - e, G)$; therefore

$$\sum_{F-e \in [F-e]} \; \sum_{(F-e)' \cong F-e} a(F - e) = \sum_{F-e \in [F-e]} s(F - e, G) a(F - e).$$

For each particular subgraph $(F - e)'$ of G, the number of true copies of F containing $(F - e)'$ in G is $b(F - e)'$. On the other hand, for each given true copy F' of F in G, F' contains exactly $\varepsilon(\tilde{F})$ subgraphs of the form $F' - e$, where $e \in \tilde{F}$. Therefore

$$\sum_{F-e \in [F-e]} \; \sum_{(F-e)' \cong F-e} b(F - e)' = s(F, G)\varepsilon(\tilde{F})$$

and hence

$$t' = \sum_{F-e \in [F-e]} [s(F - e, G) a(F - e)] - s(F, G)\varepsilon(\tilde{F}).$$

Note that $s(F,G)\varepsilon(\overline{Q})$ counts the total number of true copies of F appearing in $\{G+e \mid e \in \overline{Q}\}$ which is $t-t'$. Hence

$$s(F,G)\varepsilon(\overline{Q}) = t(F) - \sum_{F-e\in[F-e]} [s(F-e,G)a(F-e)] + s(F,G)\varepsilon(\tilde{F}),$$

and

$$s(F,G) = \frac{[\sum_{F-e\in[F-e]} s(F-e,G)a(F-e)] - t_G(F)}{\varepsilon(\tilde{F}) - \varepsilon(\overline{Q})}.$$

Let H be any subgraph of K_ν such that $H \cap T = G \cap T$. In this case the definition of a full graph F is the same whether in reference to graph G or graph H. Also $s(F,G)$ and $s(F,H)$ refer to the number of true G-copies of F and the number of true H-copies of F, respectively. Notice that in Equation (1), substituting H for G yields the following equality.

$$s(F,H) = \frac{[\sum_{F-e\in[F-e]} s(F-e,H)a(F-e)] - t_H(F)}{\varepsilon(\tilde{F}) - \varepsilon(\overline{Q}_H)} \tag{2}$$

2.2 Applications of the Counting Formula

This section contains the proofs of two results stated in the introduction. Both proofs follow the same format, an induction based on the counting formula. As in Section 2.1, G is a fixed graph, regarded as a subgraph of K_ν, and S,T is a particular partition of the edge set of K_ν such that S is either a complete subgraph of order not more than ν or a complete bipartite subgraph of order ν. The graphs Q and $\overline{Q}$ are defined as in Section 2.1.

Let F be a subgraph of G and $G-e$ any graph in $\{G-e\}$. We often use the terminology "whether or not $e \in F$ can be *recognized*". This means that whether or not $e \in F$ can be determined by the set $\{G-e\}$. For example, let F be the complete graph K_4 and let G be a graph obtained by adding a vertex u and an edge joining u and a vertex $v \in F$. In $\{G-e\}$, there is a graph (which is in fact $G-(u,v)$) consisting of a K_4 and an isolated vertex. Therefore, from $\{G-e\}$ we know that G must consist of a complete graph K_4 and an additional edge adjacent to a vertex of K_4. Hence given any $G-e \in \{G-e\}$, $e \in F$ if and only if $G-e$ contains no vertex of degree zero. Thus for any $G-e \in \{G-e\}$, whether or not $e \in F$ can be *recognized.* Likewise, for any subgraph F' of $G-e$, we say that F' can be *recognized* in $G-e$ if, using any information in $\{G-e\}$, F' can be distinguished in $G-e$. Using the same example as above, let $F' = G-(u,v)$. Since we already know from $\{G-e\}$ that G consist of a complete graph K_4 and an additional edge adjacent to a vertex of K_4, F' can be *recognized* in $G-(u,v)$ as the unique complete subgraph of order four.

For ease in applying the counting formula, the concept of complement edge reconstruction is introduced as follows. A graph H is called a *complement edge reconstruction* of G if there is a bijection $f : E(\overline{G}) \to E(\overline{H})$ such that $G+e \cong H+f(e)$ for all $e \in E(\overline{G})$. This fact will also be denoted $\{G+e\} = \{H+e\}$. A graph G is called *complement edge reconstructible* if $G \cong H$ for every complement edge reconstruction H of G. It is easy to see that a graph is edge reconstructible if and only if its complement is complement edge reconstructible.

Theorem 2.2. Let G' be a separable graph with 3-connected pruned center P. If $\varepsilon(P) > \frac{1}{2}\binom{\nu(P)}{2}$, then G' is edge reconstructible.

Proof. It will be shown that G' is edge reconstructible by proving that $\overline{G'}$ is complement edge reconstructible. This is done by applying Theorem 2.1 to $G = \overline{G'}$. In this case $\overline{G} = G'$ is a separable graph with pruned center P. Let S be the complete graph induced by $V(P)$. Then $Q = G \cap S$ and $\overline{Q} = \overline{G} \cap S = P$. Some preliminary facts are proved first.

(1) The pruned graph and pruned center of G' are reconstructible.

To see this assume that G' contains a 1-vertex since a separable graph without a 1-vertex is known to be reconstructible [6]. The center P is not a vertex because $\varepsilon(P) > \frac{1}{2}\binom{\nu(P)}{2} \geq 0$. The pruned graph of G', and hence the pruned center of G', is reconstructible because the pruned graph of G' is exactly the pruned graph of $G' - e$, where e is any edge incident with a 1-vertex. Such a $G' - e$ can be recognized because the degree sequence is reconstructible.

(2) The set $\{G + e \mid e \in \overline{Q}\}$ can be recognized from $\{G + e\}$.

This is equivalent to showing that $\{\overline{G} - e \mid e \in P\}$ can be recognized from $\{\overline{G} - e\}$. Given $\overline{G} - e$, to determine whether or not $e \in P$ compare *pruned*$(\overline{G} - e)$ with *pruned*$(\overline{G})$. If they do not have the same number of 2-blocks, then $e \notin P$. This is because P is 3-connected. If they do have the same number of 2-blocks, the pruned centers are isomorphic if and only if $e \notin P$. Note that these comparisons are possible because the pruned graph and pruned center of G are reconstructible by paragraph (1) above.

(3) For any complement edge reconstruction H of G it may be assumed without lose of generality that $H \cap T = G \cap T$.

A *branch* B of $\overline{G}$ is a maximal subgraph of $\overline{G}$ that contains exactly one vertex of the pruned center P, called the *root* of B. A branch B is called a *rooted branch* if the root of B is labeled. (For example the root can be labeled by coloring it blue.) Hereafter, branch will always mean rooted branch.

To prove statement (3) assume, without lose of generality, that the graph $\overline{H}$ is obtained from $\overline{G} - e$, where $e \in \overline{Q}$, by adding back an edge somewhere in the pruned center of $\overline{G} - e$. Note that paragraph (2) above implies that, whether or not $e \in \overline{Q}$, is determined. Hence the set of branches of $\overline{H}$ and $\overline{G}$ are the same. Take complements to obtain $H \cap T = G \cap T$.

(4) For each full graph $F \subset K_\nu$ the number of copies $t_G(F)$ is uniquely determined by the set $\{G+e\}$.

It was shown in paragraph (2) above that $\{G+e \,|\, e \in \overline{Q}\}$ is determined by $\{G+e\}$. We now show that $t_G(F)$ is determined by $\{G+e \,|\, e \in \overline{Q}\}$. The argument in paragraph (3) above shows that the set of branches of $\overline{G}$ can be recognized from each $\overline{G}-e$ where $e \in \overline{Q}$. Therefore $G \cap T$ can be recognized in each $G+e$, which means that whether or not a subgraph F' of $G+e$ contains all the edges in $G \cap T$ and satisfies $F' \cong F$ with $G \cap T$ corresponding $G \cap T$ is determined. Also a subgraph of $G+e$ can contain no other edges of T besides $G \cap T$ because $e \in \overline{Q} \subseteq S$. Hence whether or not a subgraph of $G+e$ is a copy of F can be recognized.

Let H be a complement edge reconstruction of G. We must show that H and G are isomorphic. For any full graph $F \subseteq G$, we have $\varepsilon(\tilde{F}) = \varepsilon(F \cap Q_G) \leq \varepsilon(Q_G) < \frac{1}{2}\binom{\nu(P)}{2} < \varepsilon(P) = \varepsilon(\overline{Q}_G) = \varepsilon(\overline{Q}_H)$, the last equality because $\overline{Q}_G = P \cong \overline{Q}_H$. Therefore the denominators in Equations (1) and (2) of Section 2 are not zero. It will be proved by induction on $\varepsilon(F \cap Q_G)$ that $s(F,H) = s(F,G)$. When $\varepsilon(F \cap Q_G) = 0$, we have $s(F,H) = s(F,G) = 1$ since $H \cap T = G \cap T$ by paragraph (3) above. Assume that $s(F,H) = s(F,G)$ for all full $F \subseteq G$ with $\varepsilon(F \cap Q) < k$. Then for F with $\varepsilon(F \cap Q) = k$, we have $s(F-e,H) = s(F-e,G)$ for all $F-e \in [F-e]$ by the induction hypothesis. Also $t_G(F)$ is uniquely determined by the set $\{G+e\}$ by paragraph (4) above, hence $t_H(F) = t_G(F)$. Theorem 2.1 now yields $s(F,H) = s(F,G)$ for any full $F \subseteq G$. Take $F = G$ to obtain $s(G,H) = s(G,G) \geq 1$. Hence $G \cong H$, that is, $G = \overline{G'}$ is complement edge reconstructible.

REMARK. In the proof of Theorem 2.2, all that is actually assumed about the pruned center is that $P - e$ is 2-connected for any edge $e \in P$. In this sense the condition that P be 3-connected can be reduced.

Theorem 2.3. If $G' = (A, B)$ is a 2-edge-connected bipartite graph, then G' is edge-reconstructible if $\varepsilon(G') > \frac{1}{2}\nu(A)\nu(B)$.

Proof. We will prove that $\overline{G'}$ is complement edge reconstructible by applying Theorem 2.1 to $G = \overline{G'}$. In this case $\overline{G} = (A, B)$ is a bipartite graph. Let S be the complete bipartite graph consisting of all edges between A and B. Then $Q = G \cap S$ and $\overline{Q} = \overline{G} \cap S = \overline{G}$. Denote $m = \nu(A)$ and $n = \nu(B)$. Let H be a complement edge reconstruction of G, that is $\{H + e\} = \{G + e\}$. For any $e \in \overline{G}$ the bipartite graph $\overline{G} - e$ is connected. Therefore the bipartition (A, B) of $\overline{G}$ is determined by $\overline{G} - e \in \{\overline{G} - e\}$. Hence any reconstruction $\overline{H}$ of $\overline{G}$ must be bipartite. Without loss of generality, it may be assumed that H is obtained by adding an edge to $\overline{G} - e$, and hence H must have the same bipartition as $\overline{G}$. Therefore $\overline{G} \cap T = \overline{H} \cap T = \emptyset$, which implies that $H \cap T = G \cap T$. Let F be any full subgraph of G. Notice that $\varepsilon(\tilde{F}) = \varepsilon(F \cap Q_G) \leq \varepsilon(Q_G) = \varepsilon(G \cap S) < \frac{mn}{2} < \varepsilon(G') = \varepsilon(\overline{Q}_G) = \varepsilon(\overline{Q}_H)$, the last equality because $\overline{Q}_G = \overline{G}$, $\overline{Q}_H = \overline{H}$ and $\varepsilon(\overline{G}) = \varepsilon(\overline{H})$. Therefore the denominators in Equations (1) and (2) of Section 2 are not zero.

We next show by induction on $\varepsilon(F \cap Q)$ that $s(F, H) = s(F, G)$ for every full subgraph F of G. When $\varepsilon(F \cap Q) = 0$, we clearly have $s(F, H) = s(F, G) = 1$. Assume that $s(F, H) = s(F, G)$ for all full F in G with $\varepsilon(F \cap Q) < k$. Then for any full graph $F \subseteq G$ with $\varepsilon(F \cap Q) = k$, the induction hypothesis implies that $s(F - e, H) = s(F - e, G)$ for all $F - e \in [F - e]$. To show $s(F, H) = s(F, G)$ using Theorem 2.1, we need only show that $t_G(F) = t_H(F)$. Since $\{H + e\} = \{G + e\}$, this is equivalent to showing that the total number of copies $t_G(F)$ of F in the set $\{G + e \mid e \in \overline{Q}_G\}$ is uniquely determined by $\{G + e \mid e \in \overline{Q}\} = \{G + e\}$ (recall $\overline{Q} = \overline{G}$). Since the bipartition (A, B) can be recognized in each $\overline{G} - e$, the two cliques induced by A and B (whose union is $T = G \cap T$) can be recognized in each $G + e$. Therefore, whether or not a subgraph F' of $G + e$ contains $G \cap T$, and whether or not $F' \cong F$

with $G \cap T$ corresponding to $G \cap T$, can be recognized. That is, whether or not F' is a copy of F can be recognized. Therefore, $t_G(F)$ is uniquely determined.

It has been shown that $s(F, H) = s(F, G)$. Take $F = G$ to obtain $s(G, H) = s(G, G) \geq 1$. Hence $G \cong H$, that is, $G = \overline{G'}$ is complement edge reconstructible.

CHAPTER 3
THE MAIN RESULT

3.1 A Sketch of the Proof of the Main Theorem

Because the proof of the main theorem is somewhat involved, we begin with a sketch.

Main Theorem Let G' be a graph with pruned center P. If $\varepsilon(P) > \frac{1}{4}p(p+3)$, then G' is edge reconstructible.

The notation $\overline{G}$, instead of G, is used in this section to denote a separable graph, because the counting formula will be applied to the complement of G and this will make the notation consistent with Section 3.4. The counting formula counts the number $s(F,G)$ of a certain type of subgraph of G isomorphic to a given graph F. Using the counting formula, it will be shown that, for any edge reconstruction $\overline{H}$ of $\overline{G}$ and any given subgraph F of G, $s(F,G) = s(G,H)$. Taking $F = G$, we get $s(G,H) = s(G,G) = 1$, which implies that $G \cong H$ and hence $\overline{G} \cong \overline{H}$, thus proving the main theorem.

In order to apply the counting formula, however, certain facts must be verified. This is where technical difficulties arise. A subgraph $\overline{Q}$ of $\overline{G}$ must be defined with the following properties. (1) For each $\overline{G} - e$, whether or not $e \in \overline{Q}$ can be recognized. (2) $\overline{Q} - e$ can be recognized in $\overline{G} - e$ when $e \in \overline{Q}$. (3) $\overline{Q}$ has sufficiently many edges so that the denominator of the counting formula is not zero. To show (1), it is necessary to prove that the set of branches of G is edge reconstructible. This is implied by Theorem 3.2 in Section 3.2. To guarantee that (2) holds, $\overline{Q}$ has to be chosen as a very special subgraph of P. If we simply choose $\overline{Q} = P$ as in Section 2.2 for the case

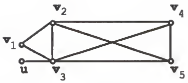

Figure 3.1. A separable graph $\overline{G}$.

of 3-connected pruned center P, then it is possible that $\overline{Q} - e$ is not recognizable in $\overline{G} - e$. For example, in Figure 3.1, P is the subgraph induced by $\{v_1, \cdots, v_5\}$. The edges (v_1, v_3) and (u, v_3) are indistiguishable in $\overline{G} - e$, where $e = (v_1, v_2)$. Hence it cannot be recognized whether $\overline{Q} - e$ is $\overline{G} - \{v_1, u\} \cup (v_1, v_3)$ or $\overline{G} - \{v_1, u\} \cup (u, v_3)$. In fact the situation can be much more complicated. In Figure 3.2, P is the subgraph induced by vertices $\{1, \cdots, 27\}$. Let B_1 be the subgraph of P induced by $1, \cdots 5$, excluding the edge $(4, 5)$. If we simply choose $\overline{Q}$ to be P in this example, then in $G - e$, $B_1 - e$ and the subgraph induced by $u_1, u_2, u_3, 27$, which is a branch of $\overline{G}$, are indistinguishable. Further, if we try to avoid this problem by choosing $\overline{Q}$ to be $P - \{1, 2, 3\}$, then in $G - e'$, B_1 and $B_2 - e'$ are indistinguishable, where B_2 is the subgraph of P induced by $7, \cdots 10$ excluding the edge $(9, 10)$. Thus the vertices $6, 7, 8$ have to be excluded from $\overline{Q}$ too. Similarly, the vertices $11, \cdots 13, 16, \cdots 18, 21, \cdots 23$ have to be excluded from $\overline{Q}$. The precise definition of $\overline{Q}$ is complicated. The primary step, the construction of a subgraph C, called *core* of P, is discussed in Section 3.3. Lemma 8 in Section 3.3 proves that $\overline{Q}$ has sufficiently many edges so that the denominator of the counting formula is not zero.

3.2 The Reconstruction of the Set of Branches

In 1969, Greenwell and Hemminger [5] proved that, for a separable graph G, the set β_G is reconstructible if there is a 1-vertex v such that $G - v$ has at least two branches with 1-vertices. To obtain the main results in Section 6, Greenwell and Hemminger's result on the reconstruction of β_G must be extended. This is done in

Theorem 3.2 below. *Throughout this section, a 2-block is always non-trivial unless otherwise stated.*

Lemma 3.1. Let G be a 2-connected graph, not a cycle, and let $u, v \in V(G)$. Then either (1) there is a vertex $w \neq u, v$ such that $G - w$ contains at most one 1-vertex and exactly one 2-block and u is in this 2-block; or (2) G is obtained from a cycle containing v by adding an edge between the two neighbors of v.

Proof. It is not hard to show that any 2-connected graph G can be constructed in stages: $G_0 \subset G_1 \subset \ldots \subset G_m = G$, where G_0 is a cycle containing u and v, and $G_{i+1} = G_i \cup p_{i+1}$ for a path p_{i+1} where $G_i \cap p_{i+1}$ is exactly the endvertices of p_{i+1}. Moreover G_i is 2-connected for each i. Consider two cases.

Case 1. All paths $p_1, p_1, \ldots p_m$ are single edges. Then G consists of a single cycle G_0 with chords. A chord divides the cycle G_0 into two arcs A and B; say $u \in A$. Call a chord *minimal* if there exists no other chord with both endvertices on B. Since G is not a cycle there exists a minimal chord. Choose a minimal chord and choose w on the corresponding arc B adjacent to an endvertex of B. This can always be done such that condition (1) of the lemma is satisfied unless there is exactly one minimal chord such that its corresponding arc B has length 2 and vertex v is the interior vertex of B, in which case the condition (2) of the lemma is satisfied.

Case 2. One of the paths $p_1, p_1, \ldots p_m$ is not a single edge. Let k be the largest integer such that p_k is a path of length greater than 1. Then $p_{k+1}, p_{k+2}, \cdots, p_m$ are

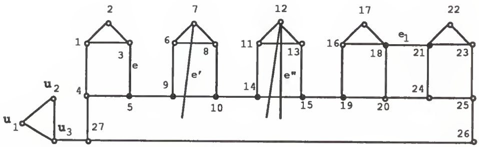

Figure 3.2. Part of the graph $\overline{G}$. Some branches are not shown here.

all single edges. Some of these edges may be chords of path p_k. A chord subtends an arc B on path p_k. As before call such a chord *minimal* if there exists no other chord with both endvertices on B. Choose w on an arc B corresponding to a minimal chord, adjacent to an endvertex of B. (If p_k has no chords, choose w adjacent to an endvertex of p_k.) This vertex $w \neq u, v$ satisfies the conditions of the lemma.

Remark The conclusion (2) in Lemma 3.1 implies that there is a $w \neq u, v$ such that $G - w$ is a path. In fact, w can be chosen as either neighbor of v. Note also that $G - v$ is a cycle containing u.

A 2-block that has only one cutpoint in *pruned*(G) will be called an *end-block* of G. *Pruned*(G) and the pruned center P of G are clearly reconstructible from $G - x$, where x is any 1-vertex of G. The next result states that, excepting exceptional circumstances, the set of branches of graph is reconstructible.

Theorem 3.2 The set of branches of G is reconstructible unless all the following hold: (1) G has exactly two branches, (2) one branch contains all the 1-vertices of G, (3) the other branch contains exactly one end-block, and (4) the pruned center of G is a vertex or an edge.

Proof. Assume that G contains a 1-vertex; otherwise G is known to be reconstructible [6. Take a $G - x \in \{G - x\} = \{H - x\}$ such that x is a 1-vertex. Such a $G - x$ can be recognized from $\{G - v\}$ since the degree sequence is reconstructible.

The proof of the reconstructibility of the set of branches is now divided into somewhat technical cases. When we say a case is "recognizable," we mean that the class of graphs covered by this case is recognizable, that is, the property determining the case is reconstructible. We will not prove the recognizability for easy cases.

Case 1. G contains exactly one 1-vertex x.

This case is recognizable since the degree sequence is reconstructible. Let the vertex adjacent to x be r. Without loss of generality, assume that r is a vertex on *pruned*(G); otherwise the unique 1-vertex of $G - x$ must be r, and G can be reconstructed from $G - x$ by adding a vertex x joined to r.

Case 1.1. G contains exactly one 2-block.

In this case *pruned*(G) $= P$, and the graph G has only one branch, which is the edge rx.

Case 1.2. G contains more than one 2-block and r is on an end-block, denoted D, of *pruned*(G) but r is not the cutpoint of D in *pruned*(G).

Assume that *pruned*(G) has at least three 2-blocks. Otherwise, the case is excluded from Theorem 3.2. To show this case is recognizable, it is first shown that, from the graph $G - r$, which can be recognized in $\{G - v\}$ as the only graph that contains an isolated vertex, whether r is in an end-block and is not a cut-point of *pruned*(G), is recognizable. First, r is a cut-point of *pruned*(G) if and only if $G - r$ has more than one non-trivial (that is, not an isolated vertex) connected component. Hence, whether or not r is a cut-point of *pruned*(G) is recognizable. Next, assuming that r is not a cut-point of *pruned*(G), we will show that, whether or not r is in an end-block, is recognizable from $G - r$. Define a matching process between the end-blocks of *pruned*(G) and the end-blocks of *pruned*($G - r$) as follows. Match each end-block of *pruned*(G) with an isomorphic end-block of *pruned*($G - r$). There are possibly end-blocks of *pruned*($G - r$) with tree-growth rooted on a vertex which is not a cut-point in *pruned*($G - r$). A *tree-growth* is a maximal connected subgraph that has exactly one vertex in common with *pruned*(G)). Exclude these end-blocks from the matching process. The matching starts from the largest end-block of *pruned*(G) and proceeds to the smallest one. Observe the following two facts: (1) If r is not in any end-block, then all end-blocks of *pruned*(G) are matched to end-blocks of *pruned*($G - r$), although *pruned*($G - r$) may have end-blocks left unmatched. (2) If r is from an end-block D', then, during the matching process defined above, there will be an end-block $D'' \cong D'$ (possibly $D'' = D'$), such that there is no end-block of *pruned*$(G - r)$ left to match D''. From the above observations, it can be seen that, whether or not r is on an end-block, is recognizable. Hence Case 1.2. is recognizable.

The above argument also shows that the 2-block D is reconstructible. In fact, for the graph $G - r$, where r is on an end-block but r is not a cutpoint of *pruned*(G), the first unmatched end-block D' of *pruned*(G) must be isomorphic to D.

Let u be the cutpoint of D in *pruned*(G). We now show that D, with u and r labeled, is reconstructible. Let $D_1 \neq D$ be an end-block of G and let u_1 be the cutpoint on D_1. By the remark following Lemma 3.1, there is a $w \in D_1$, $w \neq u_1$, such that $D_1 - w$ contains only one 2-block, or $D_1 - w$ is a path. There are four possibilities.

(i) $D_1 - w$ contains only one 2-block which contains u_1.

First, it is showed that we can choose a w' so that $D_1 - w'$ contains no path or contains a path of length at least two if we do not insist that $D_1 - w'$ contain only one 2-block. If $D_1 - w$ contains no *pendant edge* (which is an edge incident with a 1-vertex), then choose w to be w'. Assume that $D_1 - w$ contains a pedant edge. If $d_G(w) \neq 2$, we can choose the 1-vertex w_1 of this pendant edge as w'. If $d_G(w) = 2$, then let $w_2 \neq w_1$ be the other neighbor of w. In $G - w_2$, w is a 1-vertex on a path of length two. We can choose w_2 as w' if there is no path of length one in $G - w_2$. Otherwise let w_3 be a 1-vertex on a path of length one in $G - w_2$ and choose w_3 as w'. Now $G - w'$ contains no pendant edge. So in $G - w'$, the only end-block with a pendant edge rooted on it must be D. Hence consider a $G - w'$ that has only one end-block containing a root of a pendant edge, and that end-block has exactly $\nu(D)$ vertices. This end-block must be D. Hence D, with u and r labeled, is reconstructed.

(ii) $D_1 - w$ is a path of length at least two. Again, D, with u and r labeled, can be recognized from $G - w$.

Let $D_2 \neq D$ be a 2-block adjacent to D_1. Note that such a D_2 exists since *pruned*(G) has at least three 2-blocks.

(iii) $D_1 - w$ is an edge, that is, D_1 is a triangle; D_2 is not an end-block of $G - w$ or D_2 is an end-block of $G - w$ but $D_1 - w$ is rooted on the cutpoint of D_2 in *pruned*$(G - w)$.

In this case, D with u and r labeled, can be recognized form $G - w$.

(iv) $D_1 - w$ is an edge, that is, D_1 is a triangle; D_2 is an end-block of $G - w$; and $D_1 - w$ is not rooted on the cutpoint of D_2 in *pruned*$(G - w)$.

Consider the following two cases. (a) If $D_2 \cong D$, with the root of $D_1 - w$ (denoted u_1) corresponding to r, and the cutpoint of D_2 in *pruned*$(G - w)$ (denoted u') corresponding to u, then D, with u and r labeled, can be recognized from $G - w$, up to isomorphism.

(b) If the above isomorphism does not exists, then G is reconstructible, which can be proved as follows. Let H be a reconstruction of G. Without lose of generality, assume that H is obtained from $G - w$ by adding to it the vertex w and two edges incident to w. We can either let w join the two vertices of $D_1 - w$, in which case $G = H$, or let w join x and r, in which case we will prove *pruned*$(G) \ncong$ *pruned*(H) and get a contradiction since *pruned*(G) is reconstructible. Note that D is not a triangle; otherwise, since we have to have $G - r \cong H - u_1$ (which are the only graphs containing isolated vertices in $\{G - e\}$ and $\{H - e\}$, respectively), D_2 must be a triangle. This contradicts the assumption that (a) is not the case. Similarly, D_2 is not a triangle. Define a pair of *outer-blocks* (B_1, B_2) to be a subgraph of G (or H) consisting of an end-block B_1 and a 2-block B_2 (trivial or non-trivial) that contains exactly two cutpoints in *pruned*(G), one between B_1 and B_2 and the other on B_2. Two pairs of *outer-blocks* (B_1, B_2) and (B_1', B_2') are isomorphic if there is an isomorphism between (B_1, B_2) and (B_1', B_2') such that cutpoints of (B_1, B_2) correspond to the cutpoints of (B_1', B_2'). Assume the number of pairs of outer-blocks isomorphic to (D_1, D_2) in G is k. Then the number of pairs of outer-blocks isomorphic to (D_1, D_2) in H must be $k-1$ for the following reasons: (1) Since (a) is not the case, (D_1, D_2) in G is not isomorphic to (D', D) in H, where D' is the triangle xrw in H; (2) D_2 cannot be in a pair of outer-blocks in H that is isomorphic to (D_1, D_2) since D_1 is a triangle while D_2 is not. Hence we have shown that *pruned*$(G) \neq$ *pruned*(H), a contradiction. We have now shown that G is reconstructible in case (b).

Having proved that D, with r and u labeled, is reconstructible, we now proceed to show that G is reconstructible. Lemma 3.1 and the remark following it guarantee the existence of a $w \in D, w \neq u, r$ such that either $D - w$ contains exactly one 2-block which contains u or $D - w$ is a path. Therefore we can always find a connected graph $G - w$ in $\{G - v\}$ satisfying one of the following two sets of conditions: (1) The number of 2-blocks in G and $G - w$ are the same; there is a unique 2-block D' of $G - w$ such that all the tree-growths are rooted on D' and D' is a proper subgraph of D ; (2) The number of 2-blocks in $G - w$ is one less than the the number of 2-blocks in G, and there is tree-growth T in $G - w$ containing all 1-vertices. If (1) is the case, then G can be reconstructed from $G - w$ by replacing D' and all tree-growth rooted on it with D and the edge rx rooted on D. If (2) is the case, then G can be reconstructed from $G - w$ by replacing T with D and the edge rx rooted on D. In either case, G is reconstructible.

Case 1.3. G contains more than one 2-block and r is not on any end-block other than being a cutpoint of an end-block.

A vertex of a tree is *peripheral* if it is an end of a longest path. An end-block of G is a *peripheral-block* if it corresponds to a peripheral vertex of the block-cutpoint tree of *pruned*(G). Take a peripheral-block D such that D intersects the least number of other peripheral-blocks, and D has the least number of vertices among all such peripheral-blocks. Let the cutpoint on D be u, and let the number of peripheral-blocks containing u be n_u. Note that the n_u and $\nu(D)$ are reconstructible, because *pruned*(G) is reconstructible. When D is not a cycle, Lemma 3.1 and the remark following it states that there is a $w \in D$ such that $D - w$ contains only one 2-block which contains u. When D is a cycle, there is a $w \in D$ such that $D - w$ is a path . Therefore we can find a $G - w'$ in $\{G - v\}$ satisfying one of the four following conditions: (1) (corresponding to the case that D is not a cycle) $G - w'$ contains a cutpoint u' that belongs to exactly n_u peripheral-blocks, and among these peripheral-blocks there is one, denoted B, having less than $\nu(D)$ vertices; (2) (corresponding to the case that D is a cycle and $n_u \geq 2$) $G - w'$ contains a cutpoint u' that belongs to exactly $n_u - 1 > 0$ peripheral-blocks and there is a path of length $\nu(D) - 2$, denoted B, rooted on u'; (3) (corresponding to the case that D is a cycle, $n_u = 1$ and the 2-block adjacent to D is non-trivial) $G - w'$ contains a path of length $\nu(D) - 2$, denoted B, with its root u' on an end-block, but u' is not a cutpoint of this end-block in *pruned*(G); (4) (corresponding to the case that D is a cycle, $n_u = 1$, and the 2-block adjacent to D is an edge) $G - w'$ has a tree-growth T containing a path of length at least $\nu(D) - 1$.

If (1) is the case, then, since $\nu(D)$ is reconstructible, B, having fewer vertices than D, is not a peripheral-block of *pruned*(G). There is a bijection between the set of peripheral-blocks of *pruned*(G) and the set of peripheral-blocks of *pruned*$(G - w')$ such that all pairs of corresponding peripheral-blocks except one pair, denoted (D', B), are isomorphic with cut-point corresponding to cut-point under the isomorphism. Since *pruned*(G) is reconstructible, G can be reconstructed from $G - w'$, by replacing B and the path p rooted on B (if there is such a path) by D'. If (2) is the case, compare the set of 2-blocks of *pruned*$(G - w')$ with the set of 2-blocks of *pruned*(G). Let D' be the peripheral-block of *pruned*(G) that is missing from *pruned*$(G - w)$. Now G can be reconstructed from $G - w'$ by replacing B by D'. If (3) is the case, then B can always be distinguished from rx since r is not on any end-block other than being a cutpoint of an end-block. Hence G can be reconstructed from $G - w'$ by replacing B by a block D' exactly as in cases (1) and (2). Finally, if (4) is the case, let T be the unique tree-growth in $G - w'$ containing a path of length at least $\nu(D) - 1$. There is a 1-vertex y of $G - w'$ which is a peripheral vertex of T. Now G can be reconstructed by replacing a path of length $\nu(D) - 2$ containing y by a cycle of length $\nu(D)$.

Case 2. G contains more than one 1-vertex, all 1-vertices in the same branch B.

Case 2.1. *pruned*$(G) = P$, that is, G contains only one 2-block and B itself is a tree.

To show that Case 2.1. is recognizable, it is sufficient to show that the following is recognizable: all the 1-vertices are in the same branch. This can be shown as follows. If G has exactly two 1-vertices, x and y, and both $G - x$ and $G - y$ contains an edge as the unique branch, then 1-vertices are not in the same branch. Otherwise all the 1-vertices are in the same branch if and only if, in every $G - x$ where x is a 1-vertex, all 1-vertices are in the same branch.

To show that the branch B, with root labeled, is reconstructible, let the root of B be u. If P is a cycle, then G is a cactus which is reconstructible [8] . If P is not a cycle, by Lemma 3.1 and the Remark following it, there is vertex $w \in P$, such that $P - w$ contains at most one 1-vertex and contains only one 2-block which contains u. Hence there must be a $G - w$ in the set $\{G - v\}$ satisfying the following conditions: (1) $w \in P$; (2) $G - w$ contains exactly one 2-block; (3) $G - w$ contains a branch B' which is a tree-growth and contains at least two 1-vertices. (It is possible that $G - w$ contains another branch which is a path and may or may not share the same root with B'. But B' can always be recognized in $G - w'$ as the only branch containing at least two 1-vertices.) The B' in $G - w$ is the B in G. Note that the conditions (1), (2) and (3) can be recognized since P is reconstructible. From any graph $G - w$ satisfying (1), (2) and (3), the branch B can be reconstructed since $B' = B$.

Case 2.2. G contains more than one 2-block.

To show that this case is recognizable, it is sufficient to show that the property that all the 1-vertices are in the same branch is recognizable. Consider the case that G contains more then two 1-vertices, or contains exactly two 1-vertices such that one of them is adjacent to a 2-vertex. Then all the 1-vertices of G are in the same branch if and only if in each $G - x \in \{G - v\}$, where x is a 1-vertex, all the 1-vertices of $G - x$ are in the same branch. On the other hand, assume that G contains exactly two 1-vertices, denoted by x and y, and neither of them is adjacent to a 2-vertex. If both x and y are adjacent to a vertex v, not in *pruned*(G), then x and y are in the same branch. And this case can be recognized from $G - v$, since $G - v$ contains exactly two isolated vertices and *pruned*(G) = *pruned*$(G - v)$. If both x and y are adjacent to vertices of *pruned*(G), then the discussion is divided into three cases.

(1) At least one of x and y is adjacent to a vertex of the pruned center. In this case, by definition, x and y are in different branches. This case holds if and only if, for some $G - v \in \{G - v\}$ where v is a 1-vertex, there is a 1-vertex rooted on the pruned center.

In cases (2) and (3), assume that no 1-vertex is rooted on the pruned center.

(2) Both pendant edges are rooted on an end-block D, but not on the cutpoint u of D in *pruned*(G).

In this case, the two 1-vertices, x and y, are clearly on the same branch. This case can be recognized as follows.

(i) If (2) is the case, then the unique pendant edge in $G - x$ is rooted on an end-block D_x, but not on the cutpoint of D_x in *pruned*(G); in $G - y$, the unique pendant edge is rooted on an end-block, D_y, but not on the cutpoint of D_y in *pruned*(G); and $D_x \cong D_y$.

(ii) Conversely if G is a graph satisfying all conditions described in (i), then (2) is not the case for G if and only if there is w, such that $G - w$ has an isolated vertex and contains a 2-block $D' \cong D_x$, and there is a pendant edge rooted on D' but not rooted on its cutpoint in *pruned*$(G - w)$.

(3) Neither x nor y is adjacent to vertices of P, and there is no end-block D such that both x and y are adjacent to vertices of D other than its cut-point in *pruned*(G). In this case, whether or not all the 1-vertices are in the same branch, can be recognized as follows.

(i) Let x and y be in the same branch B, and let D be any end-block of G in B. Let the cutpoint of D in *pruned*(G) be u, and let $v \in D$, $v \neq u$, be a root of a pendant edge, if such a root exists in D. Lemma 3.1 and the Remark following it guarantees that there is $w \in D, w \neq u$, v, such that $D - w$ has only one 2-block which contains u, or $D - w$ is a path. $G - w$ contains no isolated vertex and, in $G - w$, all the 1-vertices are in the same branch, although the pruned center of $G - w$ and G may be different.

(ii) On the other hand, if x and y are in different branches of G, then, for any end-block D of G, by Lemma 3.1 and the remark following it, there is a $w \in D$, where w is neither the cutpoint of D in *pruned*(G), nor a vertex joining x or y, such that $G - w$ satisfies the following conditions: (a) $G - w$ contains no isolated vertex; (b) either $D - w$ has exactly one 2-block which contains the cutpoint of D in *pruned*(G), or $D - w$ is a path. For *any* such D and w, there is a pair of 1-vertices in $G - w$ that lie in different branches. (Recall that no 1-vertex is adjacent to a vertex of *pruned*(G).)

From observations (i) and (ii), we conclude that all 1-vertices of G are in the same branch if and only if there is a connected graph (no isolated vertex) $G - w \in \{G - v\}$

satisfying the following conditions: (1) w is on an end-block D of G; (2) $D - w$ has only one 2-block which contains the cutpoint of D in *pruned*(G), or $D - w$ is a path; (3) all the 1-vertices of $G - w$ are in the same branch. Hence Case 2.2 is recognizable, provided that whether or not a connected graph $G - w$ satisfies the following conditions (a) or (b) is recognizable: (a) w is on an end-block D of G, such that $D - w$ has only one 2-block which contains the cutpoint of D in *pruned*(G); (b) w is on an end-block D of G and $D - w$ is a path. Now $G - w$ satisfies (a) if and only if *block*(*pruned*($G - w$)), the block-cutpoint tree of *pruned*($G - w$), is equal to *block*(*pruned*(G)), and the set of end-blocks of $G - w$ is not equal to the set of end-blocks of G. And $G - w$ satisfies (b) if and only if *block*(*pruned*($G - w$)) $\subset$ *block*(*pruned*(G)) and the number of 1-vertices plus the number of isolated vertices in $G - w$ is at most one more than the number of 1-vertices of G. Hence whether or not a connected graph $G - w$ satisfies the condition (a) or (b) is recognizable.

This concludes the proof that Case 2.2 is recognizable. To show that the set of branches is reconstructible, it is sufficient to show that B is reconstructible, since all other branches can be reconstructed from $G - x$, where x is a 1-vertex. Case 2.2 is now divided into four subcases which can be recognized by observing a $G - x$, where x is a 1-vertex.

Case 2.2.1. B contains a peripheral-block, and G contains at least three branches or contains exactly two branches B and B_1, where B_1 contains at least two end-blocks.

Let $\{G - w \mid P_{G-w} = P\}$, where P_{G-w} is the pruned center of $G - w$. Note that whether or not $G - w$ and G have the same pruned center can be recognized by comparing the longest paths in their block-cutpoint trees and then comparing their pruned centers. For any $G - w \in \{G - w \mid P_{G-w} = P\}$, denote by B_w the the unique branch of $G - w$ containing at least two 1-vertices, if such a branch exists. Choose a w so that B_w is maximum (with respect to number of vertices) in the set

$\{B_w \mid P_{G-w} = P\}$. To show that there exists a w such that $P_{G-w} = P$ and that B_w exists, choose w as follows. Let D be a non-peripheral end-block not in B, or, in case all the end-blocks not in B are peripheral, let D be a peripheral-block not in B. Let the cut-point of D in *pruned*(G) be u and let $w \neq u$ be a vertex of D such that $D - w$ contains only one 2-block which contains u or $D - w$ is a path. For this w, $G - w$ satisfies the above requirement. Now the unique branch of $G - w$ containing at least two 1-vertices must be B, and thus B is reconstructed.

Case 2.2.2. B does not contain a peripheral-block.

In this case, G has at least three branches. Let D be peripheral-block; let u be the cut-point of D in *pruned*(G); and let $w \neq u$ be a vertex of D such that $D - w$ contains exactly one 2-block which contains u or $D - w$ is a path. That such a $G - w$ is recognizable can be proved as follows: As shown in the last part of the discussion on Case 2.2-(3)-(ii), whether or not w is in an end-block D such that $D - w$ contains only one 2-block which contains its cut-point u (in *pruned*(G)) or $D - w$ is a path is recognizable. Hence it only remains to prove that, whether or not D is peripheral, can be recognized. This is done by comparing *block*(*pruned*$(G - w)$) with *block*(*pruned*(G)) and, in case they are the same, by further comparing the set of peripheral-blocks of $G - w$ with the set of peripheral-blocks of G.

Now the branch B can be reconstructed as follows. Let the unique branch of $G - w$ containing at least two 1-vertices be B_1. Then B_1 must contain B. This fact, along with the uniqueness of B_1, can be proved by a routine check of the following two cases: (a) there are more than two branches of G containing peripheral-blocks; (b) there are exactly two branches of G containing peripheral-blocks. To locate B in B_1, let x be a 1-vertex and let $B - x$ be the unique branch of $G - x$ containing a 1-vertex. Define an embedding $f : V(B - x) \rightarrow V(B_1)$ (which is an isomorphism between the graph $B - x$ and a subgraph of B_1) such that (i) if b is the root of

$B - x$, then $f(b)$ is adjacent to a vertex not in $V(f(B-x))$; (ii) there is a vertex of $f(B-x)$ adjacent to a 1-vertex in $V(B_1) - V(f(B-x))$ and (iii) all the other vertices of $f(B-x)$ are not adjacent to any vertex not in $V(f(B-x))$. Now $f(b)$ must be the root of B. Hence B is located in B_1.

Case 2.2.3. G contains exactly two branches B and B_1, where B_1 contains exactly one end-block, and the pruned center P is a 2-block.

Let the two roots of B and B_1 be b and b_1, respectively. By Lemma 3.1 and the remark following it, there is a $G - w \in \{G - v\}$ satisfying the following condition (C0): $w \in P$, $w \neq b$, b_1, and $P - w$ contains exactly one 2-block which contains b or $P - w$ is a path. We will show that such a $G - w$ can be recognized in $\{G - v\}$. Such a $G - w$ must satisfy conditions C1 through C6 that follow. Moreover it will be shown that, first, for any $G - w' \in \{G - v\}$, $G - w'$ satisfies C0 if and only if $G - w'$ satisfies these six conditions, and second, whether or not $G - w'$ satisfies these six conditions is recognizable. This will prove that whether or not $G - w$ satisfying C0 can be recognized in $\{G - v\}$.

C1. $G - w$ has following branches: (1) a branch B' that contains at least one (non-trivial) 2-block and at least two 1-vertices; (2) a branch B_1' that does not share a common vertex with B' if the 2-block of B_1' containing the root of B_1' is non-trivial. In addition B_1' contains at least one (non-trivial) 2-block, at most one 1-vertex; (3) possibly a branch which is a (non-trivial) 2-block (the unique 2-block in $P - w$) that has its root in common with B'; (4) possibly a branch that is a path. Moreover $G - w$ can have no branch other than those mentioned.

C2. The number of (non-trivial) 2-blocks in $G - w$ is no more than the number of (non-trivial) 2-blocks in G.

C3. $0 \leq \nu(B') - \nu(B)$ and $0 \leq \nu(B_1') - \nu(B_1) < \nu(P) - 2$. (Note: The reason for $\nu(P) - 2$ is that B_1 shares a vertex with P and w is not in B_1'.)

C4. Let $k-1$ be the number of 2-blocks (trivial or non-trivial) in a chain (of 2-blocks) between P and a peripheral-block in G. In $G-w$, there is a chain C of k 2-blocks (trivial or non-trivial) containing a peripheral-block D in B'. (D is actually a peripheral-block D in B.) Let u be a vertex in C such that the distance between u and the pruned center of $G-w$ is minimum. (Here u is actually b.) Since B contains 1-vertices, it must be true that there is vertex of $V(C-u)$ adjacent to a vertex not in C.

C5. If $P-w$ contains exactly one (non-trivial) 2-block, then $u \in C$ (as described in C4) is contained in a (non-trivial) 2-block not in C.

C6. If $P-w$ contains no (non-trivial) 2-block, then (a) the number of (non-trivial) 2-blocks in $G-w$ is one less than that of G; (b) the number of 1-vertices in $G-w$ and G are equal or the number of 1-vertices in $G-w$ is one more than the number of 1-vertices in G, in which case there is a path rooted at b or b_1.

To show that the above conditions are sufficient for $G-w$ to satisfy the condition C0, we will show that if a $G-w$ satisfies the conditions C1 through C6, then $w \in P$, $w \neq b$, b_1 and $P-w$ contains exactly one 2-block which contains b, or $P-w$ is a path. Observe the following facts.

(F1) Assume that w is not in P, but $G-w$ satisfies C1 through C4. If $P_{G-w} = P$, then we have $\nu(B_1') - \nu(B_1) < 0$ when $w \in B_1$, and $\nu(B') - \nu(B) < 0$ when $w \in B$, contradicting C3. Consider the case $P_{G-w} \neq P$. If $w \in B_1$, then $\nu(B_1') - \nu(B_1) < 0$ or $\nu(B_1') - \nu(B_1) > \nu(P) - 2$, contradicting C3. If $w \in B$ and $B_1 \subset B_1'$, then B_1' must contain P (note that B_1' cannot be equal to B_1 since $P_{G-w} \neq P$), contradicting C3. If $w \in B$ and $B_1 \subset B'$, take B_1 to be the chain C and take the root of B_1 (in G) to be u. There is no vertex of $V(C-u)$ adjacent to any vertex not in C, contradicting C4.

(F2) Assume that $w \in P$ and that $G-w$ satisfies C1 through C6. If $P-w$ contains more that one (non-trivial) 2-block, then $G-w$ has more (non-trivial) 2-blocks than G does, contradicting C2. If $P-w$ contains exactly one (non-trivial) 2-block, then this 2-block must contain b because of C5. If $P-w$ contains no 2-block, then $P-w$ is a path because of C6.

We now show that, whether or not a $G-w$ satisfies C1 through C6, is recognizable. Whether a $G-w$ satisfies C1 through C4 is clearly recognizable. To show that whether a $G-w$ satisfies C5 and C6 is recognizable, it is sufficient to show that if $G-w$ satisfies C1 through C4, then whether or not $G-w$ satisfies C5 and C6 is recognizable. This can be shown as follows: As shown in (F1), w must be in P. Whether $P-w$ contains exactly one (non-trivial) 2-block or no (non-trivial) 2-block can be recognized by comparing the number of (non-trivial) 2-blocks in $G-w$ with the number of (non-trivial) 2-blocks in G. Now b can be recognized from $G-w$ as follows. In $G-w$, consider a chain C of k 2-blocks (trivial or non-trivial) containing a peripheral-block D in B'. Then b must be the vertex in C such that the distance between b and the pruned center of $G-w$ is minimum. Similarly b_1 can be recognized by considering a chain C_1 of k 2-blocks (trivial or non-trivial) containing a peripheral-block D_1 in B_1'.

Finally, to show that B is reconstructible, let the only branch containing at least two 1-vertices in $G-w$ be B'. This B' must contain B. Exactly as in Case 2.2.2, B can be located in B'.

Case 2.2.4. G contains exactly two branches B and B_1, where B_1 contains exactly one end-block, and the pruned center P is a vertex or an edge.

This is the case excluded in Theorem 3.2.

Case 3. G contains more than one 1-vertex, not all in the same branch. Since case 2 is recognizable, so is case 3.

Let $\beta' = \{B \mid B$ is a branch of some $G - y$ where y is a 1-vertex $\}$. The following is an algorithm to obtain β from β'.

Initialize m to the number of vertices in a largest graph in β'. Repeat the following procedure until $m = 1$.

If there is no branch of m vertices in β', then $m \leftarrow m - 1$.

Let B be a branch with m vertices in β'. Clearly $B \in \beta$.

Let $\beta' = \beta' - \{B_{-}y \mid y$ is $1 - vertex$ in $B\} - \{(k-1)B\}$, where k is the number of $1 - vertices$ not in B. In another words, k is the number of times that B repeats in β'.

At termination we clearly have $\beta' = \beta$.

Since the set $\{G - e\}$ determines the set $\{G - v\}$ [17] , a (vertex) reconstructible property or parameter is also edge reconstructible. Hence the following is a corollary of Theorem 3.2.

Corollary 3.3 The set of branches of G is edge reconstructible unless all the following hold: (1) G has exactly two branches, (2) one branch contains all the 1-vertices of G, (3) the other branch contains exactly one end-block, and (4) the pruned center of G is a vertex or an edge.

In the exceptional case of Theorem 3.2 or Corollary 3.3, the center P must be either a vertex or an edge. This provides the corollary that will be used in the main theorem.

Corollary 3.4 The set of branches and the pruned center P of G are edge reconstructible if P has at least three vertices.

3.3 Construction of the Core

In this section, the core C of the pruned center P is constructed. Its necessity was explained in Section 3.1. The following Lemma 3.5 will be used in the construction of C. Denote $d = \nu(D)$.

Lemma 3.5 Let P be a 2-connected graph with $\varepsilon(P) > \frac{p(p+3)}{4}$. Suppose that there exists an edge e in P such that $P - e$ contains more than one 2-block. Then there exists a unique 2-block $D = D_e$ in $P - e$ such that $d > p/2$ and $\varepsilon(D) > \frac{d(d+3)}{4}$.

Proof. The assumption $\varepsilon(P) > \frac{p(p+3)}{4}$ implies that $p \geq 6$. Note that $P-e$ is a chain of 2-blocks, $D_1, D_2, \ldots, D_m$. Assume, by way of contradiction, that $\nu(D_i) \leq p/2$ for each 2-block in this chain. In this case, we claim that $\varepsilon(D) \leq \frac{d(d+3)}{4}$, a contradiction. It is not difficult to show that $\varepsilon(P)$ is maximized when $m = 2$ and D_1, D_2 are complete graphs of order as equal as possible. In any case

$$\varepsilon(P) \leq 2\binom{p/2+1}{2} + 1 < \frac{p(p+3)}{4},$$

when $p \geq 4$. The contradiction shows that there exists a 2-block D in $P - e$ such that $d > p/2$. Such a D is unique since $d > p/2$.

Next assume, by way of contradiction, that $\varepsilon(D) \leq \frac{d(d+3)}{4}$. Again $\varepsilon(P)$ is maximized when $m = 2$, and the block other than D is a complete graph. Then

$$\varepsilon(P) \leq \frac{d(d+3)}{4} + \frac{(p-d+1)(p-d)}{2} + 1 \leq \frac{p(p+3)}{4},$$

a contradiction. This shows that $\varepsilon(D) > \frac{d(d+3)}{4}$.

For a graph P and a subgraph B of P, the notation $P - E(B)$ denotes the graph obtained by removing from P all edges of B and all the vertices of B that are not incident with any edge not in B. From now on, the vertices in $(P - E(B)) \cap B$ will be called *roots* of B.

Now define the subgraph C of P as follows. If $P - e$ contains only one non-trivial 2-block for all $e \in P$, then let $C = P$. If there is an edge $e \in P$ such that $P - e$ contains at least two non-trivial 2-blocks then, by Lemma 3.5, $P - e$ contains a 2-block $D' = D'_e$ with $\nu(D') > p/2$ and $\varepsilon(D') > \frac{1}{4}\nu(D')(\nu(D') + 3)$. For any such $D' = D'_e$, call $P - E(D')$ a *2-branch* of P. In this case, C is defined by the following somewhat involved procedure.

Let B_i and B_j be any two subgraphs of P, disjoint except at roots. We say that B_j lies *below* B_i if one of the following is true. Examples appear in Figure 3.3.

(A) There are subgraphs B^1, B^2 of B_i, where $B^1 \cup B^2 = B_i$, such that (a) there is an edge $e \in B^1$, e incident with at most one vertex in $P - E(B^1)$ and such that $B_j \cong B^1 - e$ with the the roots of B^1 corresponding to the roots of B_j under the isomorphism; (b) there is an edge $e' \in B^1 - B^2$ such that B^2 is isomorphic to B^1 with the roots of B^2 in $P - e'$ corresponding to the roots of in B^1; (c) $E(B^1) \cap E(B^2) \neq \emptyset$ or e' is incident with a vertex of B^2 not in $P - (E(B^2) \cup e')$.

(B) There is an edge $e \in B_i$, such that $B_i - e$ consists of two disjoint (except possibly share some common roots) subgraphs B_i^1 and B_i^2, such that $B_i^1 \cong B_i^2 \cong B_j$ with roots corresponding to roots under the isomorphism.

(C) There is an edge $e \in B_i$, such that $B_j \cong B_i - e$ with the roots of $B_i - e$ corresponding to the roots of B_j under the isomorphism.

Next construct a sequence $M = \{B_1, \ldots, B_n\}$ and subgraph D inductively as follows. Let B_1 be a largest 2-branch of P and $D_1 = P - E(B_1)$. Assume that, for each $j \leq i - 1$, B_j and D_j have been defined, where $D_j = D_{j-1} - E(B_j)$. Let B_i be a connected subgraph of D_{i-1} and $D_i = D_{i-1} - E(B_i)$, such that for any $j < i$, B_i does not intersect B_j except possibly at common roots; B_i does not contain any edge between two vertices of D_i; and B_i satisfies one of the following two conditions.

(1) If there is 2-branch of P that is a subgraph of D_{i-1}, then B_i is chosen as such a 2-branch of P with the largest number of vertices.

(2) Assume there is no 2-branch of P that is a subgraph of D_{i-1}. If there is a subgraph of D_{i-1} that is isomorphic to B_{i-1} (with roots corresponding roots under the isomorphism), then B_i is chosen to be such a subgraph. Otherwise B_i is chosen so that there is a B_j that lies below B_i and there do not exist s, t, $s < j < i < t$ so that B_s lies below B_t. Also B_i must satisfy one of the following three conditions.

(i) B_i is a graph as defined in (A) above.

(ii) all graphs that can be defined by (i) (for the fixed B_j) have been collected into M and B_i is a graph as defined in (B) above.

(iii) all graphs that can be defined by (i) and (ii) (for the fixed B_j) have been collected into M and B_i is a graph as defined in (C) above.

Finally, suppose n is the first natural number so that no subgraph of D_n exists that satisfies the conditions above. Let $C = D_n$ and call C the *core* of P. Note that, for each $B_j \in M$, the roots of B_j are $B_j \cap C$. Call each element of M a *multi-root-branch* of P. In particular, call elements of M defined in (2)-(i), (2)-(ii) and (2)-(iii) *type (A)*, *type (B)* and *type (C)* multi-root-branches, respectively. An important property of the core C is stated in Lemma 3.6. Denote $c = \nu(C)$.

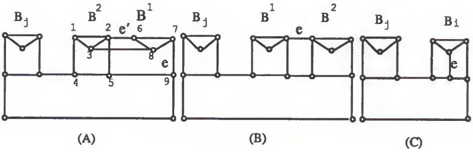

Figure 3.3.
Examples of type (A) (B) and (C) multi-root-branches. In (A), B^1 and B^2 are the subgraphs induced by vertices $2, 3, 6, 7, 8, 9$, excluding the edge $(2, 3)$, and $1, 2, 3, 4, 5, 8$, excluding the edge $(4, 5)$, respectively.

Lemma 3.6 If P is a 2-connected graph with $\varepsilon(P) > \frac{p(p+3)}{4}$, then $c = \nu(C) > p/2$ and $\varepsilon(C) > \frac{c(c+3)}{4}$.

Proof If M is empty, then $C = P$ and Lemma 3.6 is clear. Assume that M is not empty. It will first be shown that $c > p/2$. By way of contradiction, assume that $c \leq p/2$. A graph P_2 will first be constructed. Then it will be proved that $\varepsilon(P) = \varepsilon(P_2) \leq \frac{p(p+3)}{4}$, obtaining a contradiction.

The graph P_2 is constructed in two stages. The first stage is to construct a graph P_1 by performing decompositions on all multi-root-branches of P. The second stage is the construction of P_2 from P_1.

Stage 1. Perform decompositions to construct P_1.

Some definitions are introduced. Consider an edge $e = (u, v)$, where $u \notin C$ and v may or may not be in C. Define an *edge-shift* and an *edge-relocation* as follows: delete the edge (u, v) and add an edge (u, w). If $w \notin C$, then the edge e is said to be *shifted to* w and u is said to be the *fixed point* of the shift. If $w \in C$, the the edge e is said to be *relocated to* w.

Next define the decomposition. Let $B_i^1,\ B_i^2,\ B_i^3, \cdots, B_i^{n(i)}$ be subgraphs of K_p, where K_p is the complete graph of order p on the vertices of P. Let E^i be a subset of $E(K_p)$. A subgraph $B \subseteq B_i$, $B_i \in M$, is said to be *decomposed into* $B_i^1,\ B_i^2,\ B_i^3, \cdots, B_i^{n(i)}, E^i$ if all the edges of B are deleted from P and edges of $B_i^1, \cdots, B_i^{n(i)}, E^i$ are added. Note that we are only deleting and adding edges here, vertices of P remain untouched. The decomposition described below will satisfy (1) none of the $B_i^1,\ B_i^2,\ B_i^3, \cdots, B_i^{n(i)}$ contains an edge of C; (2) $B_i^1,\ B_i^2,\ B_i^3, \cdots, B_i^{n(i)}$ are pairwise disjoint except possibly at vertices of C; (3) each edge in E_i joins a vertex in B_i and a vertex in $B_j - C$ for some $B_j \in M$, $j < i$; (4) each of $B_i^1,\ B_i^2,\ B_i^3, \cdots, B_i^{n(i)}$ is isomorphic to a subgraph of a 2-branch in M.

Now perform a decomposition for each $B_i \in M$ as follows. *Throughout the remaining part of the Stage 1, B_j is always the multi-root-branch and e is always the edge as defined in (A), (B) or (C) when B_i is defined.* In the following cases (I) through (IV), either B_i or B_j is a 2-branch.

(I) If $B_i \in M$ is a 2-branch, then B_i is decomposed to itself, that is, $n(i) = 1$, $B_i^1 = B_i$ and $E^i = \emptyset$.

(II) If B_i is of type (A) and B_j is a 2-branch, then perform the following operations on B_i.

(i) As in (A), let $f : B^1 - e \cong B_j$. Shift e to a vertex in $B_j - C$ so that e joins a vertex of $B^1 - C$ to a vertex of $B_j - C$.

(ii) For each $e' = (u, v) \in B^1 - e$, where $f(v) \in C \cap B_j$, relocate e' to $f(v)$, that is, delete the edge $e' = (u, v)$ and add an edge $(u, f(v))$.

(iii) Since B^2 is isomorphic to B^1 by (A), $B^2 - B^1$ is isomorphic to a subgraph of B^1. Let $g : B^2 - B^1 \to B^1$ be an embedding. For each $e'' = (u, v) \in B^2 - B^1$, where $(g(u), g(v))$ is relocated to $f(g(v))$, relocate e'' to $f(g(v))$.

In this case, B_i is decomposed into B_i^1, B_i^2, E^i, where B_i^1 is the graph obtained from $B^1 - e$ by replacing each edge (u, v) with an edge $(u, f(v))$ as in (ii), B_i^2 is the graph obtained from $B^2 - B^1$ by replacing each edge (u, v) with an edge $(u, f(g(v)))$ as in (iii), and $E^i = \{e_1\}$, where the edge e_1 is the result of shifting e in (i).

(III) If B_i is of type (B) and B_j is a 2-branch, then perform the following operations on B_i.

(i) As in (B), $B_i - e$ consists of two disjoint subgraphs B^1 and B^2 such that $f_1 : B^1 \cong B_j$ and $f_2 : B^2 \cong B_j$. Shift the edge $e = (u, v)$, where $u \in B^1 - C$, to a vertex in $B_j - C$.

(ii) For each $e' = (u, v) \in B^1$, where $f_1(v) \in C \cap B_j$, relocate e' to $f_1(v)$, that is, delete the edge $e' = (u, v)$ and add an edge $(u, f_1(v))$.

(iii) For each $e' = (u, v) \in B^2$, where $f_2(v) \in C \cap B_j$, relocate e' to $f_2(v)$, that is, delete the edge $e' = (u, v)$ and add an edge $(u, f_2(v))$.

In this case, B_i is decomposed into B_i^1, B_i^2, E^i, where B_i^1 is the graph obtained from B^1 by replacing each edge (u, v) with an edge $(u, f_1(v))$ as in (ii), B_i^2 is the graph obtained from B^2 by replacing each edge (u, v) with an edge $(u, f_2(v))$ as in (iii), and $E^i = \{e_1\}$, where the edge e_1 is the result of shifting e in (i).

(IV) If B_i is of type (C) and B_j ($\cong B_i - e$) is a 2-branch, then shift the edge e so that e joins a vertex of $B_i - C$ to a vertex of $B_j - C$. In this case, B_i is decomposed into $B_i^1 = B_i - e$ and $E^i = \{e_1\}$, where the edge e_1 is the result of shifting e.

The decomposition is not yet complete because the case that both B_i and B_j are not 2-branches has not yet been discussed. From here proceed inductively. Assume that all $B_j \in M$, where $j \le i$, have been decomposed. In (V), (VI) and (VII), decomposition on $B_i \in M$ is performed for the cases that B_i is of type (A), (B) or (C), respectively. Before starting, we introduce a definition. Let B be a subgraph of B_i such that there is an embedding $f : B \rightarrow B_j$ for some $B_j \in M$. Suppose B_j has already been decomposed. The statement "decompose B according to the way B_j was decomposed" is defined as follows: For each edge $e \in B$ such that $f(e) \in B_j$ was shifted or relocated during the decomposition of B_j, shift or relocate the edge e to the vertex where $f(e)$ was shifted or relocated.

(V) If B_i is of type (A), then perform the following operations on B_i.

(i) As in (A), let $f : B^1 - e \cong B_j$. Shift e to a vertex in $B_j - C$ so that e joins a vertex of $B^1 - C$ to a vertex of $B_j - C$.

(ii) By the induction hypothesis, B_j has been decomposed into $B_j^1, \cdots, B_j^{n(j)}, E^j$. Note that $f : B^1 - e \cong B_j$. Decompose $B^1 - e$ according to the way B_j was decomposed. Let $B^1 - e$ be decomposed into $(B_j^1)', \cdots, (B_j^{n(j)})', (E^j)'$. For $B^2 - B^1$ (note that there is an embedding $g : B^2 - B^1 \rightarrow B^1$), decompose $B^2 - B^1$ according to the way B^1 was decomposed. Let $B^2 - B^1$ be decomposed into $(B_j^1)'', \cdots, (B_j^t)'', (E^j)''$, where $t \leq n(j)$. Then B_i is decomposed into $(B_j^1)', \cdots, (B_j^{n(j)})', (B_j^1)'', \cdots, (B_j^t)''$, $(E^j)' \cup (E^j)'' \cup \{e_1\}$, where e_1 is the edge resulted from the shifting of e in (i).

(VI) If B_i is of type (B), then perform the following operations on B_i.

(i) Note that $B_i - e$ consists of two disjoint subgraphs B^1 and B^2 such that $f_1 : B^1 \cong B_j$ and $f_2 : B^2 \cong B_j$. Shift the edge $e = (u, v)$, where $u \in B^1 - C$, to a vertex in $B_j - C$.

(ii) By the induction hypothesis, B_j has been decomposed into $B_j^1, \cdots, B_j^{n(j)}, E^j$. Since $B_j \cong B^1 \cong B^1$, decompose B^1 and B^2 according the way B_j is decomposed. Let B^1 be decomposed into $(B_j^1)', \cdots, (B_j^{n(j)})'$, $(E^j)'$, and B^2 be decomposed into $(B_j^1)'', \cdots, (B_j^{n(j)})''$, $(E^j)''$. Then B_i is decomposed into $(B_j^1)', \cdots, (B_j^{n(j)})', (B_j^1)''$, $\cdots, (B_j^{n(j)})''$, $(E^j)' \cup (E^j)'' \cup \{e_1\}$, where e_1 is the edge resulted from shifting e in (i).

(VII) If B_i is of type (C), then perform the following operations on B_i.

(i) shift the edge e so that e joins a vertex of $B_i - C$ to a vertex of $B_j - C$.

(ii) By the induction hypothesis, B_j has been decomposed into $B_j^1, \cdots, B_j^{n(j)}, E^j$. Since $B_j \cong B_i - e$, decompose $B_i - e$ according to the way B_j is decomposed. Let $B_i - e$ be decomposed into $(B_j^1)', \cdots, (B_j^{n(j)})', (E^j)'$. Then B_i is decomposed into $(B_j^1)', \cdots, (B_j^{n(j)})'$, $(E^j)' \cup \{e_1\}$, where e_1 is the edge resulted from shifting e in (i).

After all B_i $(1 \leq i \leq n)$ have been decomposed into $B_i^1, \cdots, B_i^{n(i)}, E^i$ as above, the resulting graph is denoted P_1. In the decomposition of each B_i that is not a 2-branch, the edge e is always shifted to a vertex of $B_j - C$, while any other shifted edge of B_i is shifted to a vertex of $B_l - C$, where $l < j$. Hence the edge shifts performed in the steps above shift edges incident with a common vertex to different branches. Thus no multiple edge is formed. Also, the decomposition process only involves edge-shifts and edge-relocations which do not change the number of edges in a graph. Hence the number of edges in P and P_1 are the same. The graph P_1 has the following properties: $P_1 = C \cup \{B'_1, B'_2, \cdots, B'_m\} \cup E'$, where E' is the union of E^j $(1 \leq j \leq n)$, each B'_j equals some B^l_j $(1 \leq j \leq n,\ 1 \leq l \leq n(j))$. In $P_1 - E'$, $B'_1, B'_2, \cdots, B'_m$ are disjoint except possibly at their roots; none of the $B'_1, B'_2, \cdots, B'_m$ contains an edge in C, and each B'_s, $1 \leq s \leq m$, is isomorphic to a subgraph of a 2-branch in M.

Stage 2. Construction of P_2 from P_1.

(i) Since $\nu(C) \leq p/2$ by assumption, we can partition the set of vertices of P_1 into two parts W_B and W_C, so that W_C contains exactly $\lceil p/2 \rceil$ vertices. Moreover, this can be done so that for some k, $1 < k \leq m$, W_C is the union of $V(C)$ and a (possibly empty) subset U of $V(B'_k)$ and, in case $k \neq m$, $V(B'_j)$ for all $k < j \leq m$. We have $k > 1$, since $\nu(B'_1) = \nu(B_1) < p/2$ by the definition of a 2-branch. Further the set U can be chosen so that $U \cap V(C) = V(B'_k) \cap V(C)$. Let $W_B = V(P_1) - W_C$.

From P_1, a graph P_2 will be constructed so that P_2 consists of two subgraphs, denoted C' and B', where $V(C') = W_C$ and $V(B') = W_B$ and $B' \cap C'$ is the two roots of $B_1 = B'_1$. To do this, all edges joining vertices of W_C to vertices of W_B (except those of B_1 and C) have to be shifted again. To make the next steps easier to comprehend, an example is introduced. Figure 3.4 shows a graph P with multi-root-branches $B_1, \ldots, B_n$. Figure 3.5 shows the graph P_1. The edges a', b', c', d', e' resulted from edge shifts. For simplicity, no edge relocation is considered in this example. Assume that W_C is the union of $V(C)$ and $V(B'_j)$ for all $k < j \leq m$ and $U = \{1, 4, 5\}$.

(ii) Suppose that the B'_k in (i) above is one of the subgraphs into which the multi-root-branch $B_s \in M$ was decomposed. For all edges of P_1 that are incident with vertices in $[V(B_{s+1}) \cup \cdots \cup V(B_n)] - V(C)$ or in $[V(B_s) - V(C)] \cap [U \cup V(B'_{k+1}) \cdots \cup V(B'_m)]$ and that were the result of shifting, undo these edge-shifts. Here the operation "undo an edge-shift" is defined as follows. Suppose that an edge-shift deletes edge (u, v) and

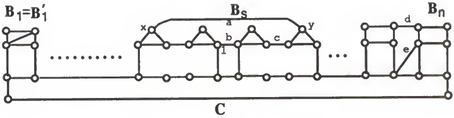

Figure 3.4. A graph P.

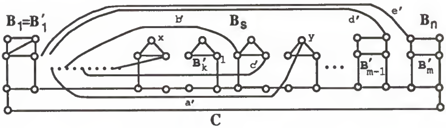

Figure 3.5. The graph P_1.

adds edge (u, w); then *undo the edge-shift* on (u, w) means delete the edge (u, w) and add back the edge (u, v). In the example, the edge-shifts for the edges a', b', c', d', e' are undone. Figure 3.6 shows the resulting graph.

(iii) After step (ii), there may be edges between vertices in $[V(B_s) - V(C)] \cap [U \cup V(B'_{k+1}) \cdots \cup V(B'_m)]$ and vertices in $[V(B_s) - V(C)] \cap [V(B'_1) \cup \cdots \cup V(B'_{k-1}) \cup V]$. Among these edges, those joining vertices of U and vertices of V will be dealt with in step (iv). All other such edges are the result of the "undo edge-shift" in step (ii) above and these are shifted again as follows. For each edge $(x, y) \notin B'_k$, where $x \in [V(B_s) - V(C)] \cap [V(B'_1) \cup \cdots \cup V(B'_{k-1}) \cup V]$ and $y \in [V(B_s) - V(C)] \cap$

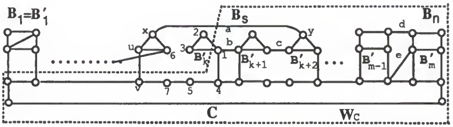

Figure 3.6. The resulting graph after undoing some edge-shifts on P_1.

$[V(B'_{k+1}) \cdots \cup V(B'_m) \cup U]$, shift (x, y) to the vertex to which (x, y) was shifted when constructing P_1, but this time, choose x instead as the fixed point. Recall that the choice of fixed point in an edge shift is arbitrary when neither of the end points of the edge is in C. For example, in Figure 3.6, edge $a = (x, y)$ is such an edge. Shift a again by deleting a and add an edge a'' between x and the vertex to which a was shifted when constructing P_1. The resulted edge a'' is shown in Figure 3.7. As explained in the last paragraph of Stage 1, since edge shifts always shift edges incident with the same vertex to different branches, no multiple edge is formed.

(iv) For each edge between a vertex u of B'_j, $1 < j < k$, and a vertex v of C, delete the edge (u, v) and add an edge between u and a vertex of B_1. See Figure 3.7. Further, for each edge between $u \in U$ and $v \in V$, delete the edge (u, v) and add an edge between v and a vertex of B_1. Since B_1, by the definition of the core, is the largest 2-branch of P and B'_k can be embedded into a 2-branch by the properties of the decomposition, we have $\nu(B'_k) \leq \nu(B_1)$. Hence no multiple edges need be formed. For example, in Figure 3.7, the edge (u, v) is deleted and an edge between u and a vertex of B_1 is added. Also the edges $(1, 2)$, $(1, 3)$ joining vertices of U to vertices of V are deleted and two edges joining vertices $\{2, 3\}$ to vertices of B_1 are added.

When the operations described in above (i)-(iv) are completed, the resulting graph is denoted P_2. Let C' be the subgraph of P_2 induced by the vertex set W_C and let B' be the subgraph of P_2 consisting of all edges of $E(P_2) - E(C')$. Notice that

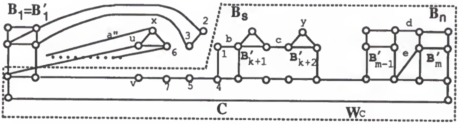

Figure 3.7. The graph P_2.

$|V(C')| = |W_C| = \lceil p/2 \rceil$ and that there are at most $\lfloor p/2 \rfloor + \lfloor p/2 \rfloor = p$ edges between the two vertices of $V(B') \cap V(C')$ and the vertices of $V(B') - V(C')$. Now

$$\begin{aligned} \varepsilon(P) &= \varepsilon(P_2) \leq \varepsilon(C') + \varepsilon(B') \\ &\leq [(p+1)/2][(p-1)/2]/2 + (p/2)(p/2-1)/2 + p < p(p+3)/4, \end{aligned}$$

which is a contradiction. Hence $\varepsilon(C) > p/2$.

We next show, by contradiction, that $\varepsilon(C) > \frac{c(c+3)}{4}$. Assume that $\varepsilon(C) \leq \frac{c(c+3)}{4}$. Define a graph P_3 as follows. Consider the graph P_1, after Stage 1 above. For each B_i', $1 \leq i \leq m$, delete each edge joining a vertex u of $B_i' - C$ and a vertex $v \in B_i' \cap C$, and add an edge between u and a vertex of $B_1 \cap C$. Note the following facts. (1) $|B_1 \cap C| = 2$ by the definition of 2-branch; (2) $|B_i' \cap C| \leq 2$ because B_i' is isomorphic to a subgraph of some 2-branch; (3) no edge of E' (the set of edges resulting from edge-shifts) joins a vertex of B_i' and a vertex of $B_1 \cap C$, since edges were shifted only to vertices not in C. Hence no multiple edges need be formed in the above construction of P_3. In the new graph P_3, let B'' be the subgraph consisting of all edges of $E(P_3) - E(C)$. Notice that $|V(P_2) - V(C)| = p - c$ and that there are at most $2(p-c)$ edges between the two vertices of $V(B'') \cap V(C)$ and the vertices of $V(B') - V(C)$. We have

$$\varepsilon(P) = \varepsilon(P_3) \leq \frac{c(c+3)}{4} + \frac{(p-c)(p-c-1)}{2} + 2(p-c) \leq \frac{p(p+3)}{4},$$

a contradiction.

3.4 The Main Theorem

This section contains the preliminary Lemma 3.7 and the proof of the main result.

Lemma 3.7 Let C be a graph of minimum degree at most 2 and $\varepsilon(C) > \frac{1}{4}c(c+3)$, where $c = \nu(C)$. Then there exists a k, $1 \leq k < c$, such that in the degree sequence

$d_C(v_1) \le d_C(v_2) \le \cdots \le d_C(v_k) \le d_C(v_{k+1}) \le \cdots \le d_C(v_c)$ of C, we have (1) $d_C(v_{i+1}) \le d_C(v_i) + 1$ for all $i < k$ and $d_C(v_{k+1}) > d_C(v_k) + 1$; (2) $k < c - 4$; and (3) the subgraph B induced by $\{v_{k+1}, v_{k+2}, \cdots, v_c\}$ satisfies $\varepsilon(B) > \frac{1}{2}\binom{\nu(B)}{2}$.

Proof. By way of contradiction, assume that (1) does not hold. Then the total degree of the graph C is at most $2+3+4+\cdots+(c-3)+(c-2)+(c-1)+(c-1)+(c-1)$ which implies that

$$\varepsilon(C) \le \frac{1}{2}[\frac{(2+c-1)(c-2)}{2} + 2(c-1)] < \frac{c(c+3)}{4},$$

contradicting the assumption. Thus (1) is proved. In addition it can be seen from the degree sequence that $k < c-4$; hence (2) holds. To show (3), consider two cases. The notation $\langle x_1, \cdots, x_i\rangle$ will be used to denote the subgraph of C induced by the vertices $x_1, \cdots, x_i$.

Case 1. $k \le \frac{c}{2}$.

Since $\varepsilon(C) > \frac{c(c+3)}{4}$, the total degree of the subgraph C is more than $\frac{c(c+3)}{2}$. The total degree of the graph $\langle v_1, \cdots, v_k\rangle$ is at most $2 + 3 + \cdots + (k+1) = \frac{k(k+3)}{2}$. As a consequence, the total number of edges between the graphs $\langle v_1, \cdots, v_k\rangle$ and $B = \langle v_{k+1}, \cdots, v_c\rangle$ is also at most $\frac{k(k+3)}{2}$. Hence the total degree of B is more than $\frac{c(c+3)}{2} - \frac{2k(k+3)}{2}$ which is greater than $\binom{\nu(B)}{2} = \frac{(c-k)(c-k-1)}{2}$. Hence $\varepsilon(B) > \frac{1}{2}\binom{\nu(B)}{2}$.

Case 2. $k > \frac{c}{2}$.

Observe the following facts.

(i)

$$d_C(v_{k+1}) + \cdots + d_C(v_c) \ge \frac{c(c+3)}{2} - \frac{k(k+3)}{2}.$$

(ii) The number of edges between graphs $\langle v_1, \cdots, v_{c-k}\rangle$ and $\langle v_{k+1}, \cdots, v_c\rangle$ is less than or equal to the total degree of the vertices $v_1, \cdots, v_{c-k}$, which is at most

$$2 + 3 + \cdots + (c-k+1) = \frac{(c-k+3)(c-k)}{2}.$$

(iii) The total number of edges between $\langle v_{c-k+1}, \cdots, v_k\rangle$ and $\langle v_{k+1}, \cdots, v_c\rangle$ is at most $\nu(\langle v_{c-k+1}, \cdots, v_k\rangle)\nu(\langle v_{k+1}, \cdots, v_c\rangle) = (2k-c)(c-k)$. Hence the total degree of the graph $B = \langle v_{k+1}, \cdots, v_c\rangle$ is at least $\frac{c(c+3)}{2} - \frac{k(k+3)}{2} - \frac{(c-k+3)(c-k)}{2} - (2k-c)(c-k)$, which is greater than $\binom{\nu(B)}{2} = \frac{(c-k)(c-k-1)}{2}$. Again $\varepsilon(B) > \frac{1}{2}\binom{\nu(B)}{2}$.

Theorem 3.8 Let G' be a graph with pruned center P. If $\varepsilon(P) > \frac{1}{4}p(p+3)$, then G' is edge reconstructible.

Proof. If P is a cycle, then G' is edge reconstructible by [5]. If $P-e$ is 2-connected for all $e \in P$, then G' is edge reconstructible by Theorem 2.2 and the remark following it. To show that G' is edge reconstructible in the remaining case, let H' be any edge reconstruction of G'. The counting formula will be used to show that $G = \overline{G'}$ is isomorphic to $H = \overline{H'}$, thus proving $G' \cong H'$.

Let C be the core of P as defined in Section 3.3. Define a partition S, T of the edges of K_ν, where K_ν is the complete graph on $\nu(G)$ vertices. If there is no vertex $v \in C$ such that $d_C(v) \leq 2$, then let S be the complete graph induced by $V(C)$ and T all the edges of K_ν not in S. If there is a vertex $v \in C$ such that $d_C(v) \leq 2$, the partition S, T is defined as follows. Consider the degree sequence of the core C:

$$d_C(v_1) \leq d_C(v_2) \leq \cdots \leq d_C(v_k) \leq d_C(v_{k+1}) \leq \cdots \leq d_C(v_c).$$

By Lemma 3.6, $\varepsilon(C) > \frac{c(c+3)}{4}$. Hence by Lemma 3.7, there exists k, $1 \leq k < c-4$, such that $d_C(v_{i+1}) \leq d_C(v_i) + 1$ for all $i < k$ and $d_C(v_{k+1}) > d_C(v_k) + 1$. Let S be the complete graph induced by the vertex set $\{v_{k+1}, v_{k+2}, \cdots, v_c\}$ and T the graph consisting of all edges of K_ν not in S.

Let $Q = G \cap S$ and $\overline{Q} = \overline{G} \cap S = G' \cap S$ be the graphs to be used in the counting formula of Section 2.1. The following four preliminary results will be required in applying the counting formula.

(1) The set $\{G + e \,|\, e \in \overline{Q}\}$ can be recognized in $\{G + e\}$.

This is equivalent to showing that $\{\overline{G} - e \,|\, e \in \overline{Q}\}$ can be recognized in $\{\overline{G} - e\}$. We first prove that if $e \in \overline{Q}$, then (a) $\beta_{\overline{G}-e} = \beta_{\overline{G}}$, where $\beta_{\overline{G}-e}$ is the set of branches of $\overline{G} - e$ and $\beta_{\overline{G}}$ is the set of branches of $\overline{G}$; and (b) $\nu(P_e) = \nu(P)$, where P_e is the pruned center of $\overline{G} - e$. For this, it is sufficient to show that $P - e$ is a non-trivial 2-block. Since $e \in \overline{Q}$ and $V(\overline{Q}) = \{v_{k+1}, \cdots, v_d\}$, e is not incident with any 2-vertex of P. Also $P - e$ cannot have more than one 2-block. Otherwise, by Lemma 3.5, $P - e$ must contain a 2-block, denoted A, with more than $p/2$ vertices and more than $\frac{\nu(A)(\nu(A)+3)}{4}$ edges. In this case, $B = P - E(A)$ would, by definition, be a 2-branch. But, by the construction of $\overline{Q}$, the edge $e \in B$, being contained in a 2-branch, could not be in $\overline{Q}$, which is a contradiction. Hence $P - e$ is a non-trivial 2-block.

At this point it is known that $\{\overline{G} - e \,|\, e \in \overline{Q}\} \subseteq \{\overline{G} - e \mid \beta_{\overline{G}-e} = \beta_{\overline{G}},\ \nu(P_e) = \nu(P)\}$. The set $\{\overline{G} - e \mid \beta_{\overline{G}-e} = \beta_{\overline{G}},\ \nu(P_e) = \nu(P)\}$ can be recognized in $\{\overline{G} - e\}$ since $\beta_{\overline{G}}$ and P are edge-reconstructible. Hence, to show that $\{\overline{G} - e \,|\, e \in \overline{Q}\}$ can be recognized in $\{\overline{G} - e\}$, it is sufficient to show that $\{\overline{G} - e \,|\, e \in \overline{Q}\}$ can be recognized in $\{\overline{G} - e \mid \beta_{\overline{G}-e} = \beta_{\overline{G}},\ \nu(P_e) = \nu(P)\}$. This will be done in two steps. First, it will be shown that for any $\overline{G} - e \in \{\overline{G} - e \mid \beta_{\overline{G}-e} = \beta_{\overline{G}},\ \nu(P_e) = \nu(P)\}$, whether or not $e \in D$, is recognizable. Second, it will be shown that for any $\overline{G} - e$ with $e \in D$, whether or not $e \in \overline{Q}$, is recognizable.

First step: For any $\overline{G} - e \in \{\overline{G} - e \mid \beta_{\overline{G}-e} = \beta_{\overline{G}},\ \nu(P_e) = \nu(P)\}$, *pruned*$(\overline{G} - e)$ and *pruned*$(\overline{G})$ have the same number of 2-blocks, since the number of 2-blocks in their sets of branches are the same and their pruned center are either both 2-blocks, or both an edge or a vertex. First we show that $e \in P$. By way of contradiction, assume that $e \notin P$. Since $\beta_{\overline{G}-e} = \beta_{\overline{G}}$, the pruned center of $\overline{G} - e$ is not P. But in this case, it can be checked that *pruned*$(\overline{G} - e)$ and *pruned*$(\overline{G})$ cannot have the same number of 2-blocks, a contradiction.

We now show that, whether or not $e \in D$, is recognizable. For any $B \in M$, a subgraph B' of $P - e$ is called a *B-subgraph* if there is an isomorphism $f : B \rightarrow B'$, such that, in B', only vertices in $f(B \cap D)$ can be adjacent to vertices of $P - e$ not in B'. Define a coloring process on $P - e$ as follows: For each successive j, $j \geq 1$, if there is a B_j-subgraph of $P - e$ all whose edges are uncolored, then color it; repeat this until there is no uncolored B_j-subgraph of $P - e$ left. The coloring process ends if and only if there is no uncolored B_n-subgraph left in $P - e$. Let M_{P-e} be the set of all colored B-subgraphs of $P - e$, for all $B \in M$. Note that the M_{P-e} obtained by the coloring process may not be unique. Since P is reconstructible (see the remark preceeding Theorem 3.2), so is M. Therefore to show that, whether or not $e \in D$, is recognizable, it is sufficient to show that $e \in D$ if and only if $M = M_{P-e}$ for all possible choices of M_{P-e}. This is done next.

Assume that $e \notin D$. Hence $e \in B_i$ for some $B_i \in M$. It will be shown that there is a particular choice of M_{P-e} such that $M \neq M_{P-e}$. By way of contradiction, assume that for any choice of M_{P-e}, we have $M = M_{P-e}$. Perform the coloring process on $P-e$ as follows. Starting form $j = 0$, for each $B_j \in M$, $j < i$ and $B_j \ncong B_i$, color each multi-root-branch in M that is isomorphic to B_j. After all these multi-root-branches are colored, there is no B_j-subgraph left in $P - e$, since $M = M_{P-e}$. Let the number of multi-root-branches in M that are isomorphic to B_i be denoted $n(B_i)$. After all B_j-subgraphs, for all $j < i$ and $B_j \ncong B_i$, have been colored, color the $n(B_i) - 1$ multi-root-branches in M that are isomorphic to B_i (note that B_i itself is no longer a B_i-subgraph in $P - e$). Continue arbitrarily to obtain an M_{P-e} by the coloring process. Since $M = M_{P-e}$, there must be a $B' \cong B_i$, such that $B' \in M_{P-e} - M$. Note that $e \notin B' \subset P - e$ and that B' does not intersect any multi-root-branch that was colored before B', except possibly sharing some common roots (since the elements of M_{P-e} can only intersect each other at roots). In particular, because of the ordering on the multi-root-branches, B' is not contained in any multi-root-branch of the form $B^1 \cup B^2 \cup e$, where $B^1 \cong B^2 \cong B_i$, as defined in (A) in Section 3.3. The fact that B' is not a multi-root-branch in P implies (a) e is incident with a vertex of B' other than a vertex in $P - E(B')$; or (b) $E(B') \cap E(B_i) \neq \emptyset$. But in each case $B' \cup B_i \cup e$ is a multi-root-branch, a contradiction. Note that one may argue that the fact that B' is not a multi-root-branch in P could be due to the possibility that B' intersects a multi-root-branch B_l, where $l > i$ and $B_l \ncong B_i$. But, in this case, according to the order defined in Section 3.3, (2) (in the construction of D), B' or $B' \cup B_i \cup e$ should be in M and B_l should not be in M, a contradiction.

Conversely assume that $e \in D$. For any M_{P-e} defined by the coloring process, it will be proved that $M_{P-e} = M$. We first show that, for any $B_i \in M$, B_i must be colored as a B_i-subgraph in $P - e$. By induction, it is sufficient to show that, under the assumption that for any $B_j \in M$, $j < i$ and $B_j \ncong B_i$, B_j is colored as a B_j-subgraph, then B_i must also be colored as a B_i-subgraph. By way of contradiction, assume that B_i is not colored as a B_i-subgraph. Then there must be an already colored subgraph $B \subset P - e$, such that $E(B) \cap E(B_i) \neq \emptyset$. Note that B is not in M since no two multi-root-branches can share a common edge. Also B does not intersect any multi-root-branch that appears before B_i (except possibly sharing some common roots), since these are already colored. In particular, because of the ordering on M, B is not contained in any multi-root-branch of the form $B^1 \cup B^2 \cup e$ (where $B^1 \cong B_i$) as defined in (A) or (B) in Section 3.3. Now, $B \cong B_j$ for some $j \leq i$. If $j < i$, then e must be incident with at least one vertex of B other than its roots; otherwise B would be in M. But then $B + e$ should be a type (C) multi-root-branch as defined in Section 3.3, unless it is a subgraph of a type (A) or (B) multi-root-branch as defined in Section 3.3. In either case e is not in D, a contradiction. If $j = i$, then $B \cup B_i \cup e$ (or $B \cup B_i \cup e'$, where $e' \in B_i - B_j$, if e is not incident with any vertex of B_j other than its root), is a type (A) multi-root branch, contradicting the facts that $e \in D$ and that B is not contained in any multi-root-branch of the form $B^1 \cup B^2 \cup e$ (type (A)) as defined in Section 3.3. Now that it has been shown that all multi-root-branches are colored, we next show that any B-subgraph B', that was colored in $P - e$, must be in M. By way of contradiction, assume that $B' \notin M$. Note that B' does not intersect any multi-root-branch in M, except possibly sharing some common roots, since they are all in M_{P-e}. So e must be incident with at least one vertex of B' other than its roots; otherwise B' would be in M. But in this case, $B' + e$ is already a type (C) multi-root-branch of P unless $B' + e$ is a subgraph of a type (A) or (B) multi-root-branch of P. In either case e is not in D, a contradiction. Thus we have proved that $M = M_{P-e}$. We have shown that $e \in D$ if and only if $M = M_{P-e}$ for all possible choice of M_{P-e}. Hence, whether or not $e \in D$, is recognizable.

Second step: We show that for any $\overline{G} - e$ with $e \in D$, whether or not $e \in \overline{Q}$, is recognizable. Since $V(\overline{Q}) = \{v_{k+1}, \cdots, v_d\}$ and $d_D(v_{k+1}) > d_D(v_k) + 1$, the edge e is from $\overline{Q}$ if and only if the first k terms of the degree sequences of $D - e$ and D are the same. Hence whether $e \in \overline{Q}$ is recognizable.

The argument above for proving, whether or not $e \in D$ is recognizable, also shows that if $e \in D$, then the uncolored part of $P - e$ is exactly $D - e$. Therefore $D - e$ is recognizable in $P - e$. Since $\overline{Q} - e$ is recognizable in $D - e$, as was just pointed out above, we have also proved the following result:

(1′) $\overline{Q} - e$ is recognizable in $P - e$.

Turning to the second preliminary result.

(2) For the graph $G = \overline{G'}$ and $H = \overline{H'}$, it may be assumed, without lose of generality, that $H \cap T = G \cap T$.

Take any $\overline{G} - e \in \{\overline{G} - e \mid e \in \overline{Q}_G\}$ and assume that $f : \overline{G} - e \cong \overline{H} - e'$. Since by (1), whether or not $e \in \overline{Q}$, is recognizable, we have $e' \in \overline{Q}_H$. Since by

(1′), $\overline{Q}_G - e$ is recognizable in $\overline{G} - e$ and $\overline{Q}_H - e'$ is recognizable in $\overline{H} - e'$, we have $f : \overline{Q}_G - e \cong \overline{Q}_H - e'$. Therefore if, for each $v \in \overline{G} - e$, we assign the same label to both $v \in \overline{G} - e$ and $f(v) \in \overline{H} - e'$, then we have $E(\overline{G}) - E(\overline{Q}_G) = E(\overline{H}) - E(\overline{Q}_H)$ as labeled edges. Take complements to obtain $H \cap T = G \cap T$.

(3) For each full graph F the number of copies $t_G(F)$ is uniquely determined by the set $\{G + e\}$.

It has been shown in paragraph (1) above that $\{G + e \,|\, e \in \overline{Q}\}$ is determined by $\{G + e\}$. It is now sufficient to show that $t_G(F)$ is uniquely determined by $\{G + e \,|\, e \in \overline{Q}\}$. It has been proved in (1′) that $\overline{Q} - e$ can be recognized in $P - e$ for each $e \in \overline{Q}$. Since $P - e$ remains 2-connected as previous showed in (1), $P - e$ can be recognized in $\overline{G} - e$. Hence $\overline{Q} - e$ can be recognized in $\overline{G} - e$. Taking the complement, we conclude that $Q + e$ can be recognized in $G + e$. Therefore $G \cap T = G - E(Q + e)$ can be recognized in $G + e$, which means that, whether or not a subgraph F' of $G + e$ contains all the edges in $G \cap T$, can be recognized. Also a subgraph of $G + e$, in particular F', can contain no other edges of T besides $G \cap T$ because $e \in \overline{Q} \subseteq S$. Hence, whether or not F' is full (that is, F' contains all edges in $G \cap T$, some edges of S and no other edges), can be recognized. Furthermore, whether or not F' satisfies $F' \cong F$ with $G \cap T$ corresponding to $G \cap T$ under the isomorphism, can also be recognized. Hence whether or not a subgraph F' of $G + e$ is a copy of F can be recognized.

(4) For any full graph $F \subseteq G$, the denominators in counting formulas

$$s(F, G) = \frac{[\sum_{F-e \in [F-e]} s(F - e, G)a(F - e)] - t_G(F)}{\varepsilon(\tilde{F}) - \varepsilon(\overline{Q}_G)}$$

$$s(F, H) = \frac{[\sum_{F-e \in [F-e]} s(F - e, H)a(F - e)] - t_H(F)}{\varepsilon(\tilde{F}) - \varepsilon(\overline{Q}_H)} \qquad (2)$$

are not zero.

Since $\varepsilon(\widetilde{F}) = \varepsilon(F \cap Q_G) \leq \varepsilon(Q_G)$, it is sufficient to show that $\varepsilon(\overline{Q}) > \frac{1}{2}\binom{\nu(\overline{Q})}{2}$. The last inequality follows from Lemma 3.7 by taking $\overline{Q}$ to be the B in Lemma 3.7.

We are now ready to apply the counting formula to show that H and G are isomorphic. For any full graph $F \subseteq G$, it will be proved by induction on $\varepsilon(F \cap Q_G)$ that $s(F, H) = s(F, G)$. When $\varepsilon(F \cap Q_G) = 0$, we have $s(F, H) = s(F, G) = 1$ since $H \cap T = G \cap T$ by paragraph (2) above. Assume that $s(F, H) = s(F, G)$ for all full $F \subseteq G$ with $\varepsilon(F \cap Q) < j$. Then for any F with $\varepsilon(F \cap Q) = j$, we have $s(F - e, H) = s(F - e, G)$ for all $F - e \in [F - e]$ by the induction hypothesis. Also $t_G(F)$ is uniquely determined by the set $\{G + e\}$ by paragraph (3) above, hence $t_H(F) = t_G(F)$. Equations (2) now yield $s(F, H) = s(F, G)$ for any full $F \subseteq G$. Take $F = G$ to obtain $s(G, H) = s(G, G) \geq 1$. Hence $G \cong H$, that is, $G' \cong H'$. Thus G' is edge reconstructible.

CHAPTER 4
A NEW DIRECTION FOR FUTURE STUDY

In this chapter a conjecture is posed and proved to be equivalent to the Edge-Reconstruction Conjecture. This provide a new approach to the edge-reconstruction problem.

Conjecture. Let $G \subseteq K_\nu$ be a graph with $\nu \geq 10$ and $4 \leq \varepsilon(G) \leq \nu(log_2\nu - 1)$. Then there is a number i with $\varepsilon(G) \leq i < \varepsilon(\overline{G})$ such that for any graph $J \subseteq K_\nu$ with $\varepsilon(J) = i$, the number of subgraphs of K_ν containing G and isomorphic to J is edge reconstructible.

Theorem 4.1. The conjecture is equivalent of the Edge-Reconstruction Conjecture.

Proof. It is obvious that the Edge-Reconstruction Conjecture implies the above conjecture. We need only prove that if the above conjecture is true, then any graph $G' \subseteq K_\nu$ with $\varepsilon(G') \geq 4$ is edge reconstructible. Since all the graphs with at most nine vertices are known to be reconstructible and hence edge reconstructible ([5], p.229), we may assume that $\nu \geq 10$. Moreover, if $\varepsilon(G') > \nu(log_2\nu - 1)$, then G' is edge reconstructible [19]. Hence we may assume that $\varepsilon(G') \leq \nu(log_2\nu - 1)$. Let $T = \emptyset$ and $S = K_\nu$. Then $Q = G$, $\overline{Q} = \overline{G}$ and for any subgraph $F \subseteq K_\nu$, F is always full and $\tilde{F} = F - (G \cap T) = F$. Thus Equation (1) becomes

$$s(F, G) = \frac{[\sum_{F-e \in [F-e]} s(F-e, G)a(F-e)] - t_G(F)}{\varepsilon(F) - \varepsilon(G')}. \qquad (3)$$

By assumption there is a number i with $\varepsilon(G') \leq i < \varepsilon(\overline{G'})$ such that for any graph $\overline{F}$ with $\varepsilon(\overline{F}) = i$, the number of subgraphs of K_ν containing G' and isomorphic to

$\overline{F}$ is edge reconstructible. Hence for any graph F with $\varepsilon(F) = \binom{\nu}{2} - i$, the number of subgraphs of $G = \overline{G'}$ isomorphic to F is complement edge reconstructible. Since $\varepsilon(G') \leq \varepsilon(\overline{F}) = i < \varepsilon(\overline{G'})$, we have $\varepsilon(G') = \varepsilon(\overline{\overline{G'}}) < \varepsilon(F)$, so the denominator in Equation (3) is not zero. For any graph F with $\varepsilon(F) \geq \binom{\nu}{2} - i$, we will prove $s(F, G) = s(F, H)$ by induction on $\varepsilon(F)$. In the initial case $\varepsilon(F) = \binom{\nu}{2} - i$, we have $s(F, G) = s(F, H)$ by the assumption in the conjecture. For any graph F with $\varepsilon(F) > \binom{\nu}{2} - i$, since $\varepsilon(F) > \varepsilon(G')$, the denominator of Equation (3) is not zero in each step of the induction proof. Take $F = G$ to obtain $s(G, G) = s(G, H) \geq 1$. Hence $G \cong H$, that is, $G' \cong H'$. Thus G' is edge reconstructible.

CHAPTER 5
CONCLUSION

A new counting formula for edge-reconstruction is introduced in Section 2.1 of this thesis. Using this counting formula, we have proved the following results:

(I) A 2-edge-connected bipartite graph $G = (A, B)$ is edge reconstructible if $\varepsilon(G) > \frac{1}{2}\nu(A)\nu(B)$;

(II) A separable graph G with pruned center P is edge reconstructible if $\varepsilon(P) > \frac{1}{4}p(p+3)$.

To show the latter result, we also proved the following result:

(III) The set of branches of a separable graph G is reconstructible unless all the following hold: (1) G has exactly two branches, (2) one branch contains all the 1-vertices of G, (3) the other branch contains exactly one end-block, and (4) the pruned center of G is a vertex or an edge.

This last result is an improvement on the theorem of Greenwell and Hemminger in [7].

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BIOGRAPHICAL SKETCH

Yongzhi Yang was born in 1958 in Jiangsu Province, China. He started school at age six. After graduating from high school in 1974, he worked in a food processing factory for two and half years. In 1978, he went to Beijing Institute of Technology and got his Bachelor of Science and Master of Science degree there. He then worked in a computer network company in Beijing for one year before he came to the University of Florida.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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